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FOUNDATIONS, THEORY OF SETS, LOGIC

Lyapunov, A. A. On operations on sets admitting transfinite indexes. Trudy Moskov. Mat. Obšč. **6** (1957), 195–230. (Russian)

The paper gives a complete presentation of R-sets, which are defined here in a simpler way than before [for terminology and notation see Lyapunov, Trudy Mat. Inst. Steklov. **40** (1953); MR **16**, 226]; let us remark that in the definition of ψ one reads CE_m instead E_m]. In particular, the exposition is based on the following T- and T^C-operations, respectively. A chain is defined as any set of natural numbers (0 included), and a basis N is any set of chains. If N is given, as well as a sequence of sets $\{E_n\}_n$, the corresponding δs-function $\phi_N\{E_n\}$ denotes $\bigcup_\eta \bigcap_n E_n$ $(n \in \eta \in N)$. Let N_n be a sequence of bases; the conjunctive extension of the corresponding δs-functions is defined by the basis N consisting of all the chains η such that 1) $\eta \supseteq \eta_0$ for some $\eta_0 \in N_0$, 2) if $\eta \in N$, $n \in \eta$, then N_n contains an η_n such that $\eta_n \subseteq \eta$. Let to every set E_n be associated the Ω-sequence $E_n^\xi (\xi < \Omega)$ in the following way: $E_n^1 = E_n$ (E_n^0 denotes the space), $E_n^{\alpha+1} = E_n^\alpha \cap \phi_{N_n}\{E_m^\alpha\}$, $E_n^\lambda = \bigcap E_n^\alpha$ $(\alpha < \lambda)$. Let $E_0^\Omega = \bigcap E_{0\alpha}$ $(\alpha < \Omega)$; the set E_0^Ω is determined by the given sequences E_n, N_n of sets and bases respectively; it is denoted functionally by $T_{\{N_n\}}\{E_n\}$. This "T-operation" is a δs-operation on E_n with just the basis N, i.e., $T_{\{N_n\}}\{E_n\} = \phi_N\{E_n\}$. The T^C-operation is defined in the following way. The disjunctive extension of δs-operations on E_n with bases N_n is the δs-operation with the basis M satisfying $N_0 \subseteq M$ such that if $\eta \in M$ and $n \in \eta$ then, for every $\eta_n \in N_n$, $\eta \setminus \{n\} \cup \eta_n \in M$. Putting $\hat{E}_n^1 = E_n$, $\hat{E}_n^{\alpha+1} = \hat{E}_n^\alpha \cup \phi_{N_n}\{E_m^\alpha\}$, $\hat{E}_n^\lambda = \bigcup_{\alpha<\lambda} \hat{E}_n^\alpha$, let $\hat{E}_0^\Omega = \bigcup_\alpha \hat{E}_0^\alpha$ $(\alpha < \Omega)$. Then $T^C_{\{N_n\}}\{E_n\} = \hat{E}_0^\Omega$. A sequence N_n of bases is non-chained provided there exists a sequence $\eta^0, \eta^1, \cdots, \eta^n, \cdots$ of mutually disjoint chains such that $\{0, n\} \cap \eta^n$ is empty and $\eta_n \in N_n$ implies $\eta_n \subseteq \eta^n$. Every T-operation with non-chained bases is an R-operation and vice versa. Starting with the above definitions, all the results concerning R-sets which were proved earlier by other means are now proved in a simpler way.

Đ. Kurepa (Zagreb).

Specker, Ernst. Teilmengen von Mengen mit Relationen. Comment. Math. Helv. **31** (1957), 302–314.

This review will employ the notation used in the paper of Erdös and Rado [Bull. Amer. Math. Soc. **62** (1956), 427–489; MR **18**, 458]. The author proves by ingenious combinatorial arguments that $\omega^2 \to (\omega^2, n)^{(2)}$ holds for every integer $n > 2$, but that for every integer $k > 2$ $\omega^k \to (\omega^k, 3)^{(2)}$ is false. The problem whether $\omega^\omega \to (\omega^\omega, 3)^{(2)}$ holds is left open. The author further shows that $\omega^2 \to (\omega+1, 4)^{(3)}$ does not hold. He asks the question whether if $\alpha \to (\alpha, 3)^{(2)}$ holds then $\alpha \to (\alpha, n)^{(2)}$ holds for every integer $n > 3$.

One of the interesting lemmas of the author is the following: Let $\alpha \geqq \omega^2$ be a denumerable ordinal number of the second type (i.e., a limit ordinal). Then one can map the ordinals $< \alpha$ unto the ordinals $< \omega^2$ so that the image of every sequence of type α is of type ω^2. The lemma becomes false if ω^2 is replaced by ω^3, e.g., for $\alpha = \omega^\omega$. The author leaves the following question open: For which ordinals $\alpha > \omega^3$ is it true that one can map the ordinals $\beta < \omega^3$ unto the ordinals $< \omega^3$ so that the image of every sequence of type α should be of type ω^3. *P. Erdös.*

Lorenzen, Paul. Zur Interpretation der Syllogistik. Arch. Math. Logik Grundlagenforsch. **2** (1956), 100–103.

The author shows that syllogistics without the p-modes (like Bamalip) and without weak modes (like Barbari) can be interpreted as a part of positive implicational logic augmented by two constants: 0 (falsity) and $\wedge$ (contradiction). If one adds as an axiom $(P(x) \to 0) \to \wedge$ (i.e., the principle $P \mid P$), then the remaining modes are also obtainable. *H. Hiż* (University Park, Pa.).

Lacombe, Daniel. Quelques propriétés d'analyse récursive. C. R. Acad. Sci. Paris **244** (1957), 838–840, 996–997.

The author continues his outline of a theory of recursive functions of recursive real variables [same C. R. **240** (1955), 2478–2480; **241** (1955), 13–14, 151–153, 1250–1252; MR **17**, 225, 701]. He observes, "D'une façon générale, les théorèmes d'Analyse dont la démonstration est 'constructive' se mettent trivialement sous forme 'récursive'", and offers some examples. On the other hand, a number of non-classical distinctions and restrictions appear with the introduction of recursiveness, some of which are mentioned in the present notes. Among the topics considered are sequences of functions, differentiation, and functions of bounded variation. *H. G. Rice.*

Kreisel, Georg; Lacombe, Daniel; et Shoenfield, Joseph R. Fonctionnelles récursivement définissables et fonctionnelles récursives. C. R. Acad. Sci. Paris **245** (1957), 399–402.

The functionals in this note have, as arguments, functions from natural numbers to natural numbers and, as values, natural numbers. A functional is "recursively definable" if it is defined on the set $\mathfrak{R}$ of general recursive unary functions and its value is a partial recursive function of the Gödel number of its argument.

The principal result is that any recursively definable functional is equivalent (over $\mathfrak{R}$) to a partial recursive functional in the sense of Kleene [Introduction to metamathematics, Van Nostrand, New York, 1952; MR **14**, 525]. This theorem answers in the affirmative a special case of Myhill and Shepherdson's question whether every effective operation on $\mathfrak{R}$ is a partial recursive functional on $\mathfrak{R}$ [Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039].

A definition of "recursively definable subset of $\mathfrak{R}$" is

given. This definition must have been misprinted, inasmuch as, according to it, a subset of $\mathfrak{N}$ is recursively definable if the Gödel numbers of its members from a set of natural numbers .The authors probably meant to say that a subset of $\mathfrak{N}$ is recursively definable if the Gödel numbers of its members form a general recursive set. With this suggested definition, one can prove the authors' corollary: If $\mathfrak{M}$ is a recursively definable subset of $\mathfrak{N}$, there exist disjoint sets (in fact, general recursive sets) E and F of natural numbers such that, for any general recursive function η, $\eta \in \mathfrak{M} \equiv (\mathbf{E}x)(\bar{\eta}(x) \in E)$, $\eta \notin \mathfrak{M} \equiv (\mathbf{E}x)(\bar{\eta}(x) \in F)$.

G. F. Rose (Santa Monica, Calif.).

Eichholz, Thomas. Semantische Untersuchungen zur Entscheidbarkeit im Prädikatenkalkül mit Funktionsvariablen. Arch. Math. Logik Grundlagenforsch. 3 (1957), 19–28.

Let PFK be a first-order predicate calculus in which there are n-place predicates and also n-place functions ($n=1, 2, \cdots$), and let P^1K be the calculus in which the predicates are restricted to one-place and there are no functions. A term of PFK is called improper if some or all of the variables appearing in it are bound. Let B be the set of sentences H of PFK with the property that in any n-tuple $t_1, \cdots, t_n$ which appears as an argument of some function or predicate in H, all improper terms are typographically identical. It is here shown that for any H in B it is possible to find an integer k such that 1) H is valid if and only if H is valid in every domain containing 2^k elements; 2) H is satisfiable if and only if H is satisfiable in some domain containing 2^k elements. As special cases of this result we get the decidability of P^1K and of the quantifier-free sentences of PFK. *L. N. Gál.*

See also: Quantum Mechanics: Koopman.

ALGEBRA

Rado, R. Note on independence functions. Proc. London Math. Soc. (3) 7 (1957), 300–320.

In the present note independence functions (I-functions) on a given set S are considered. In particular, if f is an I-function on S and $x_0, \cdots, x_{n-1}$ are elements of S then the relation $f(x_0, \cdots, x_{n-1})=1$ means that these elements are "independent", and $f=0$ that they are "dependent". An I-function f is said to be representable over the division ring R if there exists a mapping $x \to v(x)$ of S into some left vector space T over R such that for any n and any $x_\nu \in S$, $f(x_0, \cdots, x_{n-1})=0$ if and only if the vectors $v(x_0), \cdots, v(x_{n-1}) \in T$ are linearly dependent over R. The author deals with questions concerning representability of I-functions, which are partly related to the work of H. Whitney on this topic. In particular, a necessary condition for representability of a given I-function on an arbitrary set over at least one division ring is given and connexions between the representability of I-functions over various types of fields are investigated. There are I-functions which are not representable over any field. A special class of I-functions are those representable by a graph. In this case I-functions are defined on the set of edges, say E, of a combinatorial graph, and a subset $E' \subset E$ is dependent if and only if the graph possesses a circuit with edges in E'. Several results about such I-functions are established. The last section deals with I-functions defined on well-ordered sets S. Among all bases, i.e., maximal independent subsets of S, it is possible to single out one, the minimal basis, with certain minimal and maximal properties. The results of the author extend those of the reviewer [Práce Moravské Přírod. Společnosti 3 (1926), 37–58], J. B. Kruskal, Jr. [Proc. Amer. Math. Soc. 7 (1956), 48–50; MR 17, 1231] and others on the "shortest" spanning subgraph of a graph in which a "length" is associated with every edge. *O. Borůvka.*

★ **Séminaire Paul Dubreil et Charles Pisot, 9e année: 1955/56. Algèbre et théorie des nombres.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+213 pp. (polycopiées)

These are the notes of 19 lectures given at the Seminar of P. Dubreil and C. Pisot during the academic year 1955–1956. The Seminar included six other lectures for which no notes are included. The first and nineteenth are by J. Petresco and deal with the minimum number of defining relations for a finitely presented group. Several of the lectures deal with semi-groups. Number 10 by G. Thierrin deals with cases in which $ax=bx$ for every x implies $a=b$. Molinaro in Number 22 deals with partially ordered semigroups, and Number 21 by J. Riguet is a report on recent Russian studies, in particular a result by Liapin which says that an element x of a semi-group S has an inverse in an appropriate extension of S if x may be canceled on the right or left in S. This contrasts with the Malcev conditions for embedding S in a group. Schützenberger in Number 15 applies the theory of semigroups to polygraphic encipherment.

Two lectures deal with the theory of rings. Aubert gives a general theory of ideals based on a definition of ideals as sets with certain closure and inclusion properties. Guerindon deals with a restricted minimum condition.

Three lectures deal with abstract algebras. Guérindon discusses subdirect unions, Jaffard an algebraic theory of filters, and Lazard gives a theory of a generalized graded system which can be regarded as a sort of generalized formal power series.

MacLane discusses the functor ext(T, G) which arises in the theory of extension of abelian groups. Gauthier gives a formal power series approach to the infinitesimal geometry of plane curves over fields of finite characteristic. Ishaq discusses eigenvalues of matrices. Herz gives an algebraic approach to the theory of relays in electrical circuits.

Four lectures deal with the Theory of Numbers. Descombes improves a theorem of J. W. S. Cassels on one-sided inequalities in diophantine approximation. Poitou treats simultaneous approximations of two irrationals, and in another lecture summarizes the Roth improvement of the Thue-Siegel theorem. Chabanty discusses canonical forms for real quadratic forms.

Marshall Hall, Jr. (Columbus, Ohio).

Combinatorial Analysis

Goluběv, V. A. On the numbers $\binom{m}{n}$. Časopis Pěst. Mat. 82 (1957), 216–217. (Czech)

The author investigates expressions of the function x^n as sum of functions $\binom{x+m}{k}$, where k, m, n are non-negative integers. *Author's summary.*

Hoffman, A. J. Generalization of a theorem of Konig. J. Washington Acad. Sci. **46** (1956), 211–212.

The theorem of D. Konig [Math. Ann. **77** (1916), 453–465] states that if R is a set of qk elements and T and U are two partitions of R into q subsets, each containing k elements, then there exists a subset C of R containing exactly one element from each subset in T, and likewise in U.

Now let R have any number of elements $p_1, \cdots, p_n$, and let $S=\{S_1, \cdots, S_m\}$ be any family of subsets of R. Let A be the incidence matrix with $a_{ij}=1$ if $p_j \in S_i$ and $a_{ij}=0$ if $p_j \notin S_i$, and let α be any real number with $0<\alpha<1$. Then the theorem proved in the present article states that if the matrix A has the property that every square sub-matrix has determinant 0, 1, or -1, then there exists a subset C of R such that $|\overline{C \cap S_i}-\alpha\overline{S}_i|<1$ $(i=1, 2, \cdots, m)$ where $\overline{T}$ denotes the number of elements in a set T.

Herrera, Robert B. The number of elements of given period in finite symmetric groups. Amer. Math. Monthly **64** (1957), 488–490.

Let $N(k; r)$ be the number of elements of order r in the symmetric group of degree n. It is shown that, with the convention $N(0; 1)=1$,

$$N(k; r)=\sum_s \{(k-1)!/(k-s)!\}\sum_b N(k-s; b),$$

where $[s, b]=r$. *L. Moser* (Edmonton, Alta.).

Narayana, T. V.; and Mohanty, S. G. A note on difference equations and combinatorial identities arising out of coin tossing problems. J. Indian Soc. Agric. Statist. **7** (1955), 81–86.

Les auteurs étudient les séries S_N de "pile ou face" obtenues en jetant $2N$ fois une pièce de monnaie, lorsque, (i) pile (P) et face (F) apparaissent en tout N fois chacune dans la série, (ii) le $(2k+1)$-ième essai amène toujours P si les $2k$ essais précédents ont amené k fois P et k fois F.

Soit n le nombre des F consécutives dans la partie terminale de S_N et r le nombre des jets qui ont amené F après P. (Exemple, pour $S_N=PPFPPFFF$, on a $N=4$, $n=3$, $r=2$.) On donne l'équation aux différences de (N, n, r), nombre des séquences S_N pour lesquelles N, n et r ont des valeurs données. Chaque séquence S_N qui contient r changements de P en F correspond à une "domination" d'une partition de N en r termes non nuls par rapport à une autre, et réciproquement. [Pour les définitions voir Venkata Narayana, C. R. Acad. Sci. Paris **240** (1955), 1188–1189; MR **17**, 14; et aussi l'article analysé ci-dessous]. Solution de l'équation aux différences dans le cas des k-dominations. Relation avec divers jeux de "pile ou face" déjà étudiés par Narayana [J. Indian Soc. Agric. Statist. **6** (1954), 139–146; MR **18**, 423]. Identités. *A. Sade* (Marseille).

Narayana, T. V. A combinatorial problem and its application to probability theory. I. J. Indian Soc. Agric. Statist. **7** (1955), 169–178.

Let a_i, L_i $(i=1, \cdots, k)$ and r be integers. The author considers the problem of partitioning sets of size a_i into r non-empty sets $t_j^{(i)}$ $(j=1, \cdots, r)$ in such a manner that $t_1^{(i)}+t_2^{(i)}+\cdots+t_j^{(i)}+L_j \geqq t_1^{(i+1)}+t_2^{(i+1)}+\cdots+t_j^{(i+1)}$ for all $1 \leqq i<k$ and $1 \leqq j \leqq r$. He obtains a kth order determinant for the number of partitions satisfying the above conditions. The method is similar to that of an earlier paper [C. R. Acad. Sci. Paris **240** (1955), 1188–1189; MR **17**, 14]. Some applications to coin tossing problems are given. *A. Dvoretzky* (Princeton, N.J.).

See also: Algebra: Rado. **Elementary Algebra:** Zeckendorf.

Elementary Algebra

Zeckendorf, E. Les suites récurrentes à 2 termes. Bull. Soc. Roy. Sci. Liège **25** (1956), 574–584.

The sequences of the title are those of the form $(\cdots, t_{-1}, t_0, t_1, \cdots)$ such that t_x satisfies the recurrence $t_x=nt_{x-1}+mt_{x-2}$. When $t_0=0$, $t_1=1$, the sequence is designated by $F_x \equiv F_x(m, n)$, the F as a reminder of Fibonacci since $F_x(1, 1)$ are the ordinary Fibonacci numbers. It is shown that

$$F_x=\sum_{s=0}\binom{x-1-s}{s}m^s n^{x-1-2s}, \quad F_{-x}=(-1)^{x-1}m^{-x}F_x$$

and that

$$t_x=t_1F_x+mt_0F_{x-1}=t_{x-r}F_{r+1}+t_{x-r-1}mF_r=\sum_{s=0}\binom{u}{s}F_{r+1}^{u-s}(mF_r)^s t_{x-ur-s}.$$

J. Riordan (New York, N.Y.).

Novák, Břetislav. Remark on polynomials with integral cofficients. Časopis Pěst. Mat. **82** (1957), 99. (Czech)

The following theorem is proved. Let $f(x)=\sum_{k=0}^{n}a_k x^k$ be a polynomial with integral coefficients, let s be an integer, and suppose $f(j) \equiv \pm 1 \pmod 6$ for $j=s-1$, s, $s+1$. Then $f(n) \equiv \pm 1 \pmod 6$ for every integer n.

Linear Algebra

Marcus, Marvin. Convex functions of quadratic forms. Duke Math. J. **24** (1957), 321–326.

Let $A_1, A_2, \cdots, A_k$ be k pairwise commutative Hermitian matrices of order n, where $n \geqq k$. For each $i=1, 2, \cdots, k$, let I_i be the interval formed by all values of $(A_i x, x)$ for $\|x\|=1$. Let f be a real-valued convex function defined on $I_1 \times I_2 \times \cdots \times I_k$. The paper discusses the maximum of $f((A_1x_1, x_1), \cdots, (A_kx_k, x_k))$, when $x_1, \cdots, x_k$ vary over all sets of k orthonormal vectors in the unitary n-space. *Ky Fan* (Oak Ridge, Tenn.).

Marcus, M.; Moyls, B. N.; and Westwick, R. Some extreme value results for indefinite Hermitian matrices. Illinois J. Math. **1** (1957), 449–457.

Let A be an indefinite Hermitian matrix of order n. For a fixed k between 1 and n, let $x_1, x_2, \cdots, x_k$ vary over all sets of k orthonormal vectors in the unitary n-space. It is shown that the extremal values of $\prod_{1 \leqq j \leqq k}(Ax_j, x_j)$ are of the form

$$\prod_{j=0}^{q-1}\left(\frac{\lambda(k_j+1)+\cdots+\lambda(k_{j+1})}{k_{j+1}-k_j}\right)^{k_{j+1}-k_j}$$

where $0=k_0<k_1<\cdots<k_q=k$ and $\lambda(1), \lambda(2), \cdots, \lambda(k)$ are certain k eigenvalues of A. There is also a result concerning the extremal values of the second elementary symmetric function of (Ax_j, x_j) $(1 \leqq j \leqq k)$. *Ky Fan.*

Marcus, M.; and Moyls, B. N. Maximum and minimum values for the elementary symmetric functions of Hermitian forms. J. London Math. Soc. **32** (1957), 374–377.

Let A be a Hermitian matrix of order n. For each k between 1 and n, let $x_1, x_2, \cdots, x_k$ vary over all sets of k

orthonormal vectors in the unitary n-space. If A is non-singular, it is proved that the linear subspace spanned by a minimizing or maximizing set $x_1, \cdots, x_k$ for the function $\prod_{i=1}^{k}(Ax_i, x_i)$ is invariant under A. If A is positive semi-definite and if $0 \leqq \sigma \leqq 1$, $1 \leqq r \leqq k$, then the maximum of the rth elementary symmetric function of $(Ax_i, x_i)^\sigma$ $(1 \leqq i \leqq k)$ is shown to be

$$\binom{k}{r} k^{-\sigma r} \left(\sum_{i=1}^{k} \lambda_i \right)^{\sigma r},$$

where $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$ are the eigenvalues of A. This paper is closely related to that of M. Marcus and J. L. McGregor, Canad. J. Math. **8** (1956), 524–531 [MR **18**, 273] and the paper reviewed above.

Ky Fan (Oak Ridge, Tenn.).

Dean, P. The spectral distribution of a Jacobian matrix. Proc. Cambridge Philos. Soc. **52** (1956), 752–755.

For the Jacobian matrix M ($m_{jj}=a_j$, $m_{j,j+1}=m_{j+1,j}=b_j$, the other elements being equal to zero) of order N, the author introduces the function $L(\mu)$ equal to the number of eigenvalues of M not exceeding μ and asserts that for large N this function can to a satisfactory approximation be considered analytic. In the present article he obtains the "spectral distribution function" $D(\mu)=dL(\mu)/d\mu$.

For Jacobian matrices of odd order, Dyson [Physical Rev. (2) **92** (1953), 1331–1338; MR **15**, 492] introduced the "characteristic function" $\Omega(x)=\int_0^\infty \ln(1+x\mu)E(\mu)d\mu$, where $E(\mu)$ is the spectral distribution function of the squares of the eigenvalues. If $\Omega(x)$ is known (even if only approximately), then $E(\mu)$ is found from the inversion formula $E(\mu)=(2\pi^2\mu)^{-1}\int_{-\infty}^{\infty} \cosh \pi\alpha \left(\int_0^\infty (x\mu)^{-\frac{1}{2}}\cos[\alpha \ln(x\mu)] \times \Omega'(x)dx\right)d\alpha$. The author shows that in order to find $D(\mu)$ it is possible to make use of Dyson's formula, where for $\Omega(x)$ we take the function

$$\Omega(x)=\lim_{N\to\infty}(1/N)\sum_{j=1}^{N}\ln[(1+a_jx)(1-g_jx)],$$

with

$$g_j=\alpha_j/[1-\alpha_{j+1}/\{1-\alpha_{j+2}/(\cdots-\alpha_{N-1}))\}\cdots\},$$
$$\alpha_i=|b_i|^2x^2/[(1+a_ix)(1+a_{i+1}x)].$$

The g_j exist if $4|b_k|^2 \leqq a_k a_{k+1}$ $(k=j, j+1, \cdots)$. These conditions are met in the problem of electrons in a disordered chain of atoms.

A. A. Nudel'man (RŽMat **1957**, no. 4904).

Pignani, T. J. On certain matrix equations. Amer. Math. Monthly **64** (1957), 573–576.

The matrix equation $AYB=C$ is studied. Set $A_1=SAT$, $B_1=UBV$, $Y_1=T^{-1}YU^{-1}$, $C_1=SCV$, where A_1, B_1 are canonical matrices of ranks a, b respectively, and S, T, U, V are invertible. Then the equation reduces to $A_1Y_1B_1=C_1$. If only the first a rows and first b columns of C are nonzero, the equation has a solution. The number of solutions of maximum rank is studied. In another theorem, a very special case of the equation $\sum A_kY_kB_k=C$ is treated. The results reported were applied elsewhere [Pignani and Whyburn, J. Elisha Mitchell Sci. Soc. **72** (1956), 1–14; MR **18**, 42] to obtain existence theorems for differential systems with special boundary conditions.

J. L. Brenner (Palo Alto, Calif.).

See also: Algebra: Séminaire Dubreil et Pisot. **Combinatorial Analysis:** Hoffman. **General Theory of Numbers:** Čulík. **Differential Geometry:** Lagally. **Mechanics of Particles and Systems:** Bradistilov. **Programming, Resource Allocation, Games:** Gale; Tucker; Goldman; Thompson.

Polynomials

See: Elementary Algebra: Novák. **Fields, Rings:** Lewis. **Functions of Complex Variables:** Redheffer. **Numerical Methods:** Markovitch.

Partial Order, Lattices

Jordan, P. Die Theorie der Schrägverbände. Abh. Math. Sem. Univ. Hamburg **21** (1957), 127–138.

Largely a summary and restatement of previous results on this subject [e.g., Jordan, Akad. Wiss. Mainz. Abh. Math. Nat. Kl. **1956**, 27–42, MR **18**, 713] but with some extension. Topics are: various forms of basic axioms; ordinal sums; free skew semi-lattices; various forms of inclusion, congruence classes thereunder, and their relation to axioms; nests; construction of new skew lattices from given ones.

P. M. Whitman.

Grätzer, G.; and Schmidt, E. T. On the Jordan-Dedekind chain condition. Acta Sci. Math. Szeged **18** (1957), 52–56.

The length of a chain C is defined as the power of the set of its distinct cuts; C is called strongly maximal if C is not a proper subchain of any chain with the same ends and likewise for every homomorphic image of C. Theorem: In a distributive lattice, all strongly maximal chains between fixed ends have the same length in this sense. It is known [G. Szász, same Acta **16** (1955), 89–91, 270; MR **17**, 120] that this theorem need not hold if the length is defined as the power of the set of elements of the chain. {In the latter connection, reference might further have been made to J. Jakubík, same Acta **16** (1955), 266–269; MR **17**, 1046.}

P. M. Whitman (Silver Spring, Md.).

Jakubík, Ján. Remark on the Jordan-Dedekind condition in Boolean algebras. Časopis Pěst. Mat. **82** (1957), 44–46. (Slovak)

If S is an infinite complete Boolean algebra which is completely distributive, then S does not satisfy the Jordan-Dedekind condition of constancy of all maximal paths joining two distinct points of S. As a matter of fact, S contains a convex sublattice which is isomorphic to the power set PN of the positive integers ordered by inclusion; and PN does not satisfy the Jordan-Dedekind condition.

Đ. Kurepa (Zagreb).

Kaplansky, Irving. Any orthocomplemented complete modular lattice is a continuous geometry. Ann. of Math. (2) **61** (1955), 524–541.

Ein Verband heißt orthokomplementär, wenn es in ihm einen involutorischen Antiautomorphismus $a \to a'$ gibt, derart, daß a' stets ein Komplement von a ist. Verf. zeigt, daß ein vollständiger, orthokomplementärer und modularer Verband (v.o.m.V.) stets eine kontinuierliche Geometrie im Sinne von v. Neumann [Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 92–100] ist. (Dies braucht nicht mehr zu gelten, wenn die Voraussetzung "orthokomplementär" zu "komplementär" abgeschwächt wird.) Der Beweis wird geführt mit Hilfe der von v. Neumann begründeten Theorie der regulären Ringe [Continuous geometry, part II, Inst. Advanced Study, Princeton, 1937], nach der jeder komplementäre modulare Verband (abgesehen von wenigen Ausnahmen, im wesentlichen den nichtdesargueschen Ebenen) isomorph ist zum Verband der Hauptrechtsideale eines geeignet gewählten regulären

Ringes. Ein Ring A mit Einselement heißt regulär, wenn für jedes $a \in A$ die Gleichung $axa=a$ nach x lösbar ist und $*$-regulär, wenn er außerdem einen involutorischen Antiautomorphismus $x \to x^*$ besitzt, für den $x=0$ aus $x^*x=0$ folgt. Ein idempotentes $e \in A$ mit $e^*=e$ nennt Verf. Projektion. Jedes Hauptrechtsideal von A wird von genau einer Projektion erzeugt. Die Hauptrechtsideale bilden einen orthokomplementären Verband (Verband der Projektionen). Hierbei ist $(eA)'=(1-e)A$. Verf. nennt A vollständig, wenn der zugehörige Verband der Projektionen vollständig ist. Zwei Projektionen $e, f \in A$ heißen äquivalent, wenn es $x \in eAf$ und $y \in fAe$ mit $xy=e$ und $yx=f$ gibt; sie heißen orthogonal, wenn $ef=fe=0$ ist. Verf. zeigt, daß A nicht unendlich viele äquivalente und paarweise orthogonale Projektionen enthalten kann, denn andernfalls enthielt A einen Ring von unendlichreihigen Matrizen, und das kann zu einem Widerspruch geführt werden. Sind die Projektionen e und f äquivalent und gilt $eA \supseteq fA$, so ist $e=f$. Hieraus schließt Verf., daß der Verband der Projektionen eines vollständigen $*$-regulären Ringes eine kontinuierliche Geometrie ist. Ein v.o.m.V. L heißt vom Typ I, wenn er ein Element a mit den beiden folgenden Eigenschaften enthält: 1. Der Verband $L(0, a)$ der zwischen 0 und a gelegenen Elemente ist distributiv, 2. aus $b \supseteq a$ und $b \cap (x \cup y)=(b \cap x) \cup (b \cap y)$ für alle $x, y \in L$ folgt $b=1$. Dagegen heißt L vom Typ II, wenn L kein $a \neq 0$ mit distributivem $L(0, a)$ enthält. Jeder v.o.m.V. ist eindeutig direkte Summe eines v.o.m.V. vom Typ I und eines solchen vom Typ II. Zu einem v.o.m.V. vom Typ II kann man stets vier unabhängige perspektiv liegende Elemente angeben, deren Vereinigung 1 ist; er ist somit nach v. Neumann darstellbar als Verband von Projektionen eines $*$-regulären Ringes, also eine kontinuierliche Geometrie. Der Beweis für die Verbände vom Typ I ist etwas schwieriger, da hier von der v. Neumannschen Theorie nicht erfaßte Ausnahmeverbände (im wesentlichen nichtdesarguessche Ebenen) auftreten können; für diese Ausnahmefälle kann man aber leicht die für die kontinuierlichen Geometrien kennzeichnenden Stetigkeitsaxiome direkt beweisen, nämlich $c \cap \bigcup a_\alpha = \bigcap (c \cap a_\alpha)$ für jede aufsteigende Kette a_α von Verbandselementen und die dazu duale Beziehung. Zusammen ergibt das den in der Überschrift benannten Satz.

J. André (Zbl **65** (1956), 18).

Vinha Novais, J. A. Introduction to Boolean algebras. Gaz. Mat., Lisboa **18** (1957), no. 66–67, 1–8. (Portuguese)

See also: Groups and Generalizations: Benado. **Topological Vector Spaces:** Amemiya and Mori.

Fields, Rings

Borevič, Z. I. On extensions without simple ramification of a regular local field. Vestnik Leningrad. Univ. **11** (1956), no. 19, 41–47. (Russian)

Let k be a field which is a finite extension of degree m over the rational p-adic field and which does not contain a non-trivial pth root of unity. Šafarevič [Mat. Sb. N.S. **20(62)** (1947), 351–363; MR **8**, 560; **12**, 1001] has given a description of all normal p-extensions of k (i.e., of those extensions whose Galois groups are p-groups). Making use of the method of Šafarevič, the author describes all normal extensions of k without simple ramification. The finite extension K/k of k is called an extension without simple ramification, if the exponent of the simple ramification of the field K/k is a power of p. Let F be a free group of rank $m+1$ with a fixed system $u_0, u_1, \cdots, u_m$ of free generators. For any natural integer f prime to p let F_f denote the normal subgroup generated in F by the elements $u_0^f, u_1, \cdots, u_m$. The group F becomes a topological group by the following convention: the full system of neighborhoods of the unit element is given by the set Ξ of all those normal subgroups of F for which, with some f, $N \subseteq F_f$ and the index $(F_f:N)$ is a power of p. By the main result of the paper the Galois group of the maximal extension without simple ramification of the field k (with the topology of Krull) is topologically isomorphic to the completion $\overline{F}$ of the group F, this $\overline{F}$ being a totally disconnected topological group. Moreover, it is proved that the lattice of all finite normal extensions of k without simple ramification is dually isomorphic to Ξ, which is a lattice with the operations $N_1 \cap N_2$ and $N_1 N_2$ ($N_1, N_2 \in \Xi$).

A. Kertész (Debrecen).

Širšov, A. I. Some theorems on immersion for rings. Mat. Sb. N.S. **40(82)** (1956), 65–72. (Russian)

A ring S is called a Σ, Ω-ring when (i) Σ is an associative ring whose elements are operators in S, and (ii) Ω is a set of relations which are satisfied by all elements of S. For example if Ω contains the relation $xy-yx=0$ $[(xy)z-x(yz)=0]$, each Σ, Ω-ring is commutative [associative]. Given Σ and Ω, let there be a smallest natural number k with the property that there exists a free Σ, Ω-ring S_k with k generators which contains a Σ, Ω-ring T generated by a countable set of free generators such that each Σ-ideal in T is the contraction of its extension in S_k; then k is called the dimension of Ω over Σ, and it is shown that an arbitrary Σ, Ω-ring can be embedded in a Σ, Ω-ring N with the property that each countable subset of N lies in a subring of N generated by k elements. Some special cases are studied.

F. J. Terpstra (Pretoria).

Amitsur, S. A. Derivations in simple rings. Proc. London Math. Soc. (3) **7** (1957), 87–112.

Let A be a simple ring and D a derivation in A. The ring of differential polynomials over A, $A[t]$, is the formal polynomial ring over A where multiplication is defined using $at - ta = D(a)$ for all a in A. The author then determines all the two-sided ideals in $A[t]$; they are generated by some unique monic polynomial $\pi(t_L)$ belonging to the center of the ring $C_L[t_L]$, where C_L is the commuting ring of the right multiplications by the elements of A in the ring of endomorphisms of the additive group of A. He also shows that a simple ring A with derivation D can be imbedded in a simple ring A^* with an inner derivation D^*, where D^* induces D on A, and the minimal such ring A^* is unique up to an isomorphism. A is said to satisfy a universal differential equation if there is an equation of the form $\alpha_0 z^{(n)} + \alpha_1 z^{(n-1)} + \cdots + \alpha_n z = 0$ ($\alpha_0 \neq 0$; $z^{(n)} = D^{(n)}$), where the $\alpha_i \in C_L$, which is satisfied by all the elements of A. He then shows that if A satisfies a universal equation and if $\pi(D)$ is the monic one of lowest degree then the set of all universal equations for A is $\pi(D)C_L(D)$. Specializing these results to inner derivations he obtains new proofs of some of the results of the reviewer about the Lie structure of simple rings.

{The author has communicated to the reviewer that there is a mistake in the computation of an element at the bottom of p. 108, but with a slight modification the proof is still valid.}

I. N. Herstein (Ithaca, N.Y.).

Lewis, D. J. Ideals and polynomial functions. Amer. J. Math. **78** (1956), 71–77.

Let K be a field with a discrete valuation, I the corresponding ring of integers, and P the corresponding prime ideal. Suppose I/P has only a finite number of elements, say q. If m is a non-negative integer, let B_m be that ideal in $I[x]$ such that a polynomial is in B_m if and only if the corresponding polynomial function takes I into P^m. Clearly, B_0 is generated by 1, while B_1 is generated by x^q-x and the constant polynomial corresponding to any element of P which is not in P^2. The author extends these results by giving a specific set of $m+1$ generators for B_m when $m>1$. An analogous result is given for polynomials in more than one variable. The proofs are elementary, but not trivial. *P. T. Bateman* (Urbana, Ill.).

See also: Algebra: Séminaire Dubreil et Pisot. **Algebras:** Suliński. **Groups and Generalizations:** Hion. **Geometries, Euclidean and Other:** Havel.

Algebras

Białynicki-Birula, A.; and Żelazko, W. On the multiplicative linear functionals on the Cartesian product of algebras. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 589–593, L. (Russian summary)

Let the cardinal number of a set A be denoted by $\bar{\bar{A}}$. A cardinal number $\mathfrak{m}$ is said to have measure $\neq 0$ if there is a countably additive measure μ defined on all subsets of a set A of cardinal number $\mathfrak{m}$ such that μ vanishes for points, $\mu(A)=1$, and μ assumes only the values 0 and 1. Let $\aleph_I$ be the smallest cardinal number of measure $\neq 0$ (if one exists). Let $\{R_t\}_{t\in T}$ be a family of algebras over a field K and let $R=\mathbf{P}_{t\in T}R_t$ be the Cartesian product of the algebras R_t. Let f_{t_0} be a multiplicative linear functional on R_{t_0} for some $t_0\in T$. The mapping on R defined by $(*)f(x)=f_{t_0}(x(t_0))$ for all $x\in R$ is obviously a multiplicative linear functional on R. Theorem: Suppose that $\bar{\bar{K}}<\aleph_I$. Then every multiplicative linear functional on R has the form (*) if and only if $\bar{\bar{T}}<\aleph_I$. The proof given fails if K is finite (see line –17 of page 591). On line –10 of page 592, read $\tilde{\mu}(\phi)$ for $\mu(\phi)$. [For similar results, see S. Mazur, Ann. Soc. Polon. Math. **19** (1946), 241; and Hewitt, Fund. Math. **37** (1950), 161–189; MR **13**, 147.] *E. Hewitt.*

Suliński, A. Some characterisation of the Brown-McCoy radical. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 357–359, XXIX. (Russian summary)

This is a continuation of the theory of abstract radicals in the spirit of Kuroš [Mat. Sb. N.S. **33(75)** (1953), 13–26; MR **15**, 194] and Amitsur [Amer. J. Math. **74** (1952), 774–786; MR **14**, 347]. The definitions and notation of the review of Kuroš' paper will be used. Let K be a class of associative rings with operators, and C_1 and C_2 a splitting of the set of simple rings in K such that $C_1\cap C_2=0$. There then exist S-radicals such that C_1 consists of semisimple S-rings and C_2 consists of radical S-rings contained between an upper and a lower S-radical. Denoting this upper radical by U the author then announces the following results without proofs. Suppose that every ring in C_1 contains a unit. Then every U-semisimple ring is a subdirect sum of simple rings, and every two-sided ideal in a U-radical ring is again a U-radical ring. This generalizes a theorem of Brown and McCoy [Amer. J. Math. **69** (1947), 46–58; MR **8**, 433]. If K is the class of all associative rings with only the integers as operators the conclusions of the above theorem imply that every ring in C_1 has a unit. Finally, if K is the class of all non-associative algebras over a fixed field, then, if every U-semisimple algebra is a subdirect sum of simple algebras, every algebra in C_1 has a unit. *A. Rosenberg.*

Suliński, A. Radicals in associative algebras of finite order. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 361–363, XXIX. (Russian summary)

We use the same definitions and notation as in the preceding review. The main results announced by the author are as follows. Let K be the class of all finite-dimensional associative algebras over a fixed field. If Z, the one-dimensional zero algebra, lies in C_2, then the upper and lower radicals determined by the splitting C_1, C_2 coincide, i.e., the splitting determines a unique radical. Suppose next that C_1 consists only of Z, then: (1) an algebra A is U-radical if and only if $A^2=A$; (2) if L is the lower radical of the splitting C_1, C_2, A is L-semisimple if and only if it is semisimple in the usual sense of not containing any nonzero nilpotent ideals; (3) the radical T that results from the property "A has a unit" is properly between U and L. The radicals less than T are then characterized in terms of certain classes of indecomposable algebras. The last theorem asserts that the only radicals in K for which two-sided ideals of radical rings are again radical rings are (a) those for which Z lies in C_2 and (b) the lower radical of any splitting C_1, C_2 in which Z is part of C_1. *A. Rosenberg* (Evanston, Ill.).

Jans, James P. On the indecomposable representations of algebras. Ann. of Math. (2) **66** (1957), 418–429.

In this paper the author gives conditions under which a finite-dimensional associative algebra A has infinitely many inequivalent indecomposable representations of degree d for infinitely many d (i.e., is of strongly unbounded type). It has been conjectured by Brauer and Thrall that (1) if the degrees of the indecomposable representations of A are bounded (bounded type) then the number of inequivalent indecomposable representations is finite (finite type), and (2) over an infinite field, the lack of a bound for the degrees of the indecomposable representations (unbounded type) implies that the algebra has strongly unbounded type. That the first conjecture is true in case the field is algebraically closed and the square of the radical is zero has been shown by T. Yoshii [Proc. Japan Acad. **32** (1956), 441–445; MR **18**, 462] who also has given necessary and sufficient conditions for an algebra to be of unbounded type under these assumptions [Osaka Math. J. **8** (1956), 51–105; MR **18**, 462]. The first conjecture is also true in the case of group algebras, for it has been shown by the reviewer [Duke Math. J. **21** (1954), 377–381; MR **16**, 794] that a group algebra over a field of characteristic $p>0$ is of finite type if and only if a p-Sylow subgroup is cyclic, and of unbounded type otherwise. The first main result of the present paper states that if A has an infinite ideal lattice then it is of strongly unbounded type. This generalizes a result of Thrall [unpublished], which in turn generalized a result of Nakayama [Proc. Imp. Acad. Tokyo **16** (1940), 285–289; MR **2**, 245], where the existence of algebras of unbounded type was first noted. Two further results obtained here by the author are: If A has a finite ideal lattice then every ideal in A is principal; if A is commutative and has a finite ideal lattice then A is the ring direct sum of polynomial

algebras and is of finite type. For the second main result the author extends a method of Thrall [Bull. Amer. Math. Soc. **53** (1947), 49–50] to associate with each ideal I of A, contained in the radical N of A, an oriented graph $\mathfrak{G}(I)$ as follows. Assuming the existence of a subalgebra complement S of N in A, S is a direct sum of simple ideals S_i ($i=1, \cdots, n$). Let e_i be a primitive idempotent in S_i. Let $P_1, \cdots, P_n$ be vertices, with an edge from P_i to P_j if and only if there is an ideal B of A, contained in N, which covers I and is such that e_iIe_j is properly contained in e_iBe_j. The second main theorem states that if for some such ideal I the graph $\mathfrak{G}(I)$ has a cycle, a vertex of order >3, or a chain that branches at each end, then the algebra is of strongly unbounded type. This extends results of Thrall announced in the above mentioned abstract, and of Brauer [ibid. **47** (1941), 684]. In the proofs it is assumed that the field is infinite, and that A/N is a direct sum of ideals each isomorphic with the field. It is pointed out that since the basic algebra has this structure [C. Nesbitt and W. M. Scott, Ann. of Math. (2) **44** (1943), 534–553; MR **5**, 89] the results hold for arbitrary A over an algebraically closed field. Considerable computation is left to the reader, especially in the proof of the second main theorem. *D. G. Higman.*

Higman, D. G. On orders in separable algebras. Canad. J. Math. **7** (1955), 509–515.

Es sei g ein Dedekindscher Ring, A eine endlichdimensionale assoziative Algebra über dem Quotientenkörper K von g, und G eine g-Ordnung in A. Für jeden 2-seitigen G-Modul M werde die 1. Cohomologiegruppe $H^1(G, M)$ als g-Modul aufgefasst; $I(M)$ sei das annullierende Ideal von $H^1(G, M)$ in g, und $I(G)$ der Durchschnitt der $I(M)$ für alle M. Satz: $I(G)$ ist $\neq 0$, wenn A separabel ist, und nur in diesem Fall. Der Verfasser weist darauf hin, dass es genau dieses Satzes bedarf, um die Resultate von Maranda [Canad. J. Math. **5** (1953), 344–355; **7** (1955), 516–526; MR **15**, 100; **19**, 529] über g-Darstellungen einer endlichen Gruppe, d.h., Darstellungen durch Automorphismen von Moduln über einem Dedekindschen Ring g, zu verallgemeinern auf g-Darstellungen einer beliebigen Ordnung G in einer separablen K-Algebra A. – Der Beweis des Satzes benützt die vom Verf. [Canad. J. Math. **7** (1955), 490–508; MR **19**, 390] angegebene Charakterisierung der Separabilität einer Algebra. *B. Eckmann.*

Albert, A. A.; and Jacobson, N. On reduced exceptional simple Jordan algebras. Ann. of Math. (2) **66** (1957), 400–417.

Let C be a (generalized) Cayley algebra over a field F of characteristic $\neq 2$, and let C_3 be the (nonassociative) algebra of all 3-rowed matrices with elements in C. If $\Gamma=\mathrm{diag}\{1, \gamma_2, \gamma_3\}$ is a diagonal matrix with nonzero diagonal elements γ_i in F, then $X \to X^J=\Gamma^{-1}\bar{X}'\Gamma$ is an involution of C_3, and the set $H=H(C_3, \Gamma)$ of all J-Hermitian matrices $X=X^J$ is an exceptional central simple Jordan algebra under the composition $X \cdot Y=\frac{1}{2}(XY+YX)$. Moreover, H is reduced, according to the following definition: $1=E_1+E_2+E_3$ for orthogonal idempotents E_i such that the space H_i of all X_i satisfying $X_i \cdot E_i=X_i$ is one-dimensional (in which case the E_i are called a basic set of idempotents). It is known that any reduced exceptional central simple Jordan algebra is isomorphic to an algebra $H(C_3, \Gamma)$. In this paper it is proved that two reduced algebras $H(C_3, \Gamma)$ and $H((C^*)_3, \Gamma^*)$ are isomorphic only in case C and C^* are isomorphic. The conditions for isomorphism between $H(C_3, \Gamma)$ and $H(C_3, \Gamma^*)$ are determined, and applied to obtain results on extensions of isomorphisms between subalgebras to automorphisms on the conjugacy of basic sets of idempotents, and on Lie algebras of type F.

It is first proved that if C is a Cayley division algebra, if

(1) $E_1=\mathrm{diag}\{1,0,0\}$, $E_2=\mathrm{diag}\{0,1,0\}$, $E_3=\mathrm{diag}\{0,0,1\}$,

and if F_1, F_2, F_3 are a basic set of idempotents in $H=H(C_3, \Gamma)$, then there exists an isomorphism $A \to A^*$ of H onto an algebra $H^*=H((C^*)_3, \Gamma^*)$ such that C is isomorphic to C^*, the E_i^* are in the form (1), and the F_i^* are matrices in $(K^*)_3$, where K^* is is a quadratic subfield of C^*. Using a decomposition of C_3 relative to K_3 (where K is a quadratic subfield of C), the fundamental result that $H(C_3, \Gamma) \cong H((C^*)_3, \Gamma^*)$ implies $C \cong C^*$ is then established. If C has divisors of zero, it is shown that $H(C_3, \Gamma) \cong H(C_3, 1)$ for any Γ. Also, if C is a division algebra, then $H(C_3, \Gamma) \cong H(C_3, \Gamma^*)$ with $\Gamma^*=\mathrm{diag}\{1, \gamma_2^*, \gamma_3^*\}$ if and only if there exist nonzero elements a_2, a_3, b_2, b_3 in C, a quadratic subfield K of C, a nonsingular matrix P in K_3, and a nonzero λ in F such that

(2) $\mathrm{diag}\{1, \gamma_2^* N(b_2), \gamma_3^* N(b_3)\} = \lambda \bar{P}' \,\mathrm{diag}\{1, \gamma_2 N(a_2), \gamma_3 N(a_3)\} P,$

where $N(x)$ denotes the norm form in C. This settles completely the question of isomorphism for reduced exceptional simple Jordan algebras.

Special results for algebras containing nonzero nilpotent elements are obtained, Also, it is proved that if reduced algebras H and H^* (i) have isomorphic associated Cayley algebras C and C^* and (ii) contain reduced simple subalgebras of degree three, then any isomorphism between the subalgebras can be extended to an isomorphism between H and H^*. A necessary and sufficient condition for the conjugacy of two ordered basic sets of idempotents in any reduced algebra H is derived; in particular, (i) any two ordered basic sets of idempotents are conjugate in a split algebra H (that is, H is a reduced algebra and the associated Cayley algebra C has divisors of zero), while (ii) there are basic sets of idempotents which are not conjugate in the algebra $H(C_3, \Gamma)$, C the Cayley division algebra over the field F of all real numbers, $\Gamma=\mathrm{diag}\{1, 1, -1\}$. The remainder of the paper is devoted to algebras over a real closed field or an algebraic number field. *R. D. Schafer* (Storrs, Conn.).

Morozov, V. V. Proof of the theorem of regularity. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 191–194. (Russian)

Let L be a semi-simple Lie algebra over the field of complex numbers, and M a nonsemi-simple maximal subalgebra of L. A new proof is given of the fact that M is regular, i.e., that it has a basis consisting of elements of a Cartan subalgebra H of L and root vectors with respect to H. In fact the author proves the slightly stronger result that M contains a Cartan subalgebra of L.

W. H. Mills (New Haven, Conn.).

See also: Fields, Rings: Amitsur.

Groups and Generalizations

★ **Coxeter, H. S. M.; and Moser, W. O. J. Generators and relations for discrete groups.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. viii+155 pp. DM 32.00.

The scope and the flavour of this work are perhaps

alike best illustrated by quoting the first two sentences of the preface. "When we began to consider the scope of this book, we envisaged a catalogue supplying at least one abstract definition for any finitely-generated group that the reader might propose. But we soon realized that more or less arbitrary restrictions are necessary, because interesting groups are so numerous." In fact, the book brings together a great deal of information on generators and relations for a wide variety of groups. For instance, it deals with the crystallographic groups, with the fundamental groups of surfaces, with the symmetric and related groups, with the projective linear groups over finite fields, and with groups on real Euclidean space generated by reflections. There is also a systematic method for checking the sufficiency of a set of relations, and considerable attention to graphical representations. The book will be invaluable to anyone who believes, as I do, that progress in Group Theory depends primarily on an intimate knowledge of a large number of special groups.

G. Higman (Oxford).

Schützenberger, Marcel-Paul. Sur une propriété combinatoire des demi-groupes libres. C. R. Acad. Sci. Paris **245** (1957), 16–18.

Let S be a finitely generated free semigroup (with identity). Denote the length of $s \in S$ by $|s|$. Two subsets W and W' of S are called commutatively equivalent with respect to n if there exists a $1-1$ mapping π of the set of all elements of length less than n in W onto the set of all elements of length less than n in W' such that $w\alpha = w\pi\alpha$ for all $w \in W$ with $|w| < n$ and for all homomorphisms α of S for which $S\alpha$ is commutative. The following proposition is proven. Suppose that $P = P^2 \subset S$ is free and there exists a positive integer k such that for each $s \in S$ one can find $s', s'' \in S$ with $|s'| + |s''| \leq k$ and $s'ss'' \in P$. Then for each positive integer n one can construct $P' = P'^2$ satisfying: (1) P' is free; (2) P' is commutatively equivalent to P with respect to n; (3) if $P's \cap P'$ is not empty, then $s \in P'$; and (4) P' intersects all the right ideals of S.

P. F. Conrad (New Orleans, La.).

Benado, Mihail. Sur la théorie générale des produits réguliers. C. R. Acad. Sci. Paris **244** (1957), 1702–1704; **245** (1957), 267–270.

In a preceding note [same C. R. **244** (1957), 1595–1597; MR **19**, 385] the author defined, for the lattice of operator subgroups of an operator group G, commutating functions q on the cartesian square of the lattice to the lattice obeying a lengthy list of axioms which cannot be reproduced here. For a given set of groups G_α form their free product F. Then form that homomorphic image of F which has as kernel the group-theoretic union of all the $q(G_\alpha, G_\alpha^*)$, where G_α^* is the group union of all the G_β $(\beta \neq \alpha)$ and q is a commutating function on the subgroup lattice of F. A typical statement in the first of these notes is that all the regular products of the G_α in the sense of Golovin [Mat. Sb. N.S. **27(69)** (1950), 427–454; MR **12**, 672; **17**, 824] are so obtained. In the second of these notes, the author lets $Z(q)$ be the largest subgroup H of a group G for which $q(G, H)$ is the one-element subgroup of G. Among other results, he announces the following: Let G be a regular product of some G_α where $q(G, G_\alpha) \leqq G_\alpha$ and again of some H_β where $q(G, H_\beta) \leqq H_\beta$. Then $G/Z(q)$ is the direct product of its images under the system of Fitting endomorphisms [Benado, Math. Nachr. **14** (1955), 213–234; MR **18**, 871] associated with the first of these regular decompositions and also of the second. These two direct products have common refinements. Proofs are to be published elsewhere.

F. Haimo.

Loonstra, F. Homomorphe Abbildungen von Gruppenerweiterungen. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 44–54.

Zu gegebenen Homomorphismen h der Gruppe B auf die Gruppe B' und k der Abelschen Gruppe A auf A' werden Erweiterungen von k durch h gesucht; d.h., Homomorphismen $g: G \to G'$, wobei G eine Erweiterung von A durch B, G' eine von A' durch B' ist, derart dass k die Beschränkung von g auf A ist und h die induzierte Abbildung der Faktorgruppe B auf B'. Es kommen nur Erweiterungen G von A durch B der folgenden Art in Frage: (1) Die Operationen von B in A führen den Kern A_0 von k in sich über, diejenigen von B_0=Kern h sogar jede Restklasse von A mod. A_0 in sich. (2) Das Faktorsystem $f(\beta_1, \beta_2)$ von B in A bleibt mod. A_0 konstant, wenn β_1 und β_2 Restklassen von B mod. B_0 durchlaufen; analog hierzu kann man "zulässige" Coketten $f(\beta_1, \cdots, \beta_p)$ und Cohomologiegruppen $H_z^p(B, A)$ beliebiger Dimension definieren, wenn Operationen von B in A gemäss (1) gegeben sind. Die zu gegebenen Operationen gehörigen und mit der Aufgabe verträglichen verschiedenen Erweiterungen G von A durch B bilden eine zu $H_z^2(B, A)$ isomorphe Gruppe. Jeder solchen Erweiterung entsprechen Operationen und ein Faktorsystem von B' in A', sowie ein Homomorphismus $g: G \to G'$ der gesuchten Art von G in die zugehörige Erweiterung G'.

B. Eckmann.

Kantz, G. Eine für die Theorie der relativ-abelschen Körper grundlegende Abelsche Operatorgruppe. Monatsh. Math. **61** (1957), 151–156.

Let G be an abelian group with a finite basis and such that the totality of its elements of finite order constitutes a cyclic group Z of order n (e.g., the units of an algebraic number field). Then the ring $R(\sigma)$ of polynomials, with rational integral coefficients, in a homomorphism σ of G into itself, such that $\sigma^p = 1$, p a prime, is a ring of operators for G. They are written as exponents. The author writes $\delta = \sigma - 1$ and $\nu = \sigma^{p-1} + \sigma^{p-2} + \cdots + 1$, so that $R(\delta) = R(\sigma)$, and proves that the $R(\delta)$-ideals (δ^{p-1}, ν) and (p, ν) are equal. He defines four subgroups of G: A, B, A_1 and B_1, as the respective totalities of elements of G such that $a^\delta \in Z$, $b^\nu \in Z$, $a_1^\delta = e$ (identity element) and $b_1^\nu = e$. He proves that (1) $A \cap B = Z$; (2) if $g \in G$, $g^\delta \in A$, then $g \in A$; (3) $G^{\delta^{p-1}} G^\nu = G^\nu G^\nu$. With these preliminary results, and the use of well-known reduction and isomorphism principles for groups, the author obtains several index relations for the groups G, A, $\cdots$. These relations are of the type employed in the class-field theory and the author describes an application in that theory. [Cf. Kantz, Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Edizioni Cremonese, Roma, 1953, pp. 131–138; MR **15**, 14.]

R. Hull (Pacific Palisades, Calif.).

Honda, Kin'ya. From a theorem of Kulikov to a problem of Kaplansky. Comment. Math. Univ. St. Paul. **6** (1957), 43–48.

Let $\mathfrak{o}$ be a set of powers of a fixed prime p. Generalizing the concept of Kulikov's basic subgroup, the author defines an $\mathfrak{o}$-basic subgroup of an abelian group G as a subgroup B such that B is a direct sum of cyclic groups of orders in $\mathfrak{o}$, B is pure in G, and G/B contains no pure subgroup of order in $\mathfrak{o}$. Any abelian group contains $\mathfrak{o}$-basic subgroups, and these are unique up to isomorphism. As an application, Kaplansky's test problem III (if F is finitely gener-

ated, then $F+G\cong F+H$ implies $G\cong H$) is solved for finite F [for a general solution see P. M. Cohn, Proc. Amer. Math. Soc. **7** (1956), 520–521; MR **17**, 1182; and E. A. Walker, ibid. **7** (1956), 898–902; MR **18**, 403].

L. Fuchs (Budapest).

Plotkin, B. I. Radical and semi-simple groups. Trudy Moskov. Mat. Obšč. **6** (1957), 299–336. (Russian)

The paper begins with a summary of known facts concerning the radical and the upper radical in a group and concerning semisimple groups. In order to generalize the radical group, the class of W-groups is defined and extensively studied. A group is called a W-group provided it has an ascending normal series each factor of which is locally nilpotent or simple. If a group has an ascending normal series with simple factors, then any two such series are isomorphic. Numerous results are obtained concerning radicals in connection with finiteness conditions of various types — such as the maximal condition for subgroups, the minimal condition for descending normal series, the condition that every abelian subgroup have finite rank, the condition of being an A_4-group. Every torsion-free radical group of finite special rank contains a subgroup having finite index and having a finite rational series. An infinite group is not simple if it possesses a local system of finite subgroups for which any two members such that one contains the other are connected by an ascending normal series. The second chapter deals with various properties preserved under lattice isomorphisms; among these results are the following. If $\mathfrak{G}$ and $\mathfrak{G}^\phi$ are lattice-isomorphic groups and if $\mathfrak{G}$ is a group of any of the following types, then $\mathfrak{G}^\phi$ is a group of the same type: (i) solvable A_4-group, (ii) radical group with finite special rank, (iii) radical R^*-group, and (iv) R^*-group and RN^*-group. If $\mathfrak{G}$ and $\mathfrak{G}^\phi$ are lattice-isomorphic groups and if $\mathfrak{G}$ is a locally nilpotent group containing at least two independent elements of infinite order, then $\mathfrak{G}^\phi$ is locally nilpotent.

R. A. Good (College Park, Md.).

Lazard, Michel. Sur les groupes nilpotents et les anneaux de Lie. Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 101–190.

An N-series of a group G is a series $G=H_1\supseteq H_2\supseteq\cdots$ such that if $x\in H_i$, $y\in H_j$ then $[x, y]\in H_{i+j}$. The lower central series is the best-known example. If G has an N-series with trivial intersection, G is called an N-group. The author associates with an N-series a certain graduated Lie ring: its additive group is the direct sum of the quotients H_i/H_{i+1} written additively, and its Lie (or "bracket") multiplication is obtained from commutation in G by postulating distributivity. The graduation is defined by calling the elements of H_i/H_{i+1} "homogeneous of degree i". This generalizes the well-known construction introduced by Magnus [J. Reine Angew. Math. **182** (1940), 142–149; MR **2**, 214] for the lower central series and by Zassenhaus [Abh. Math. Sem. Hansischen Univ. **13** (1939), 200–207] for the dimension groups modulo a prime number. The interrelations between groups, their N-series, and the corresponding Lie rings are then studied; not only are known results of Magnus, Witt, Zassenhaus, and others shown to spring from a common root, but they are carried further in a number of directions. As an example the dimension groups modulo p^k are determined for primes p and all k, and used in a study of the dimension groups modulo arbitrary integers. Some of the results have already been briefly described by the author [C. R. Acad. Sci. Paris **236** (1953), 1222–1224; MR **14**, 724]. The first part of the paper closes with the statement and discussion of a number of unsolved problems that naturally arise in the context. — In the second part the Baker-Hausdorff formula is used for the study of nilpotent groups and N-series. The author's results connecting his "typical sequences" with Philip Hall's "commutator collecting process" and the inversion of the Baker-Hausdorff formula, briefly described in two notes [ibid. **236** (1953), 36–38; **237** (1953), 1377–1379; MR **14**, 617; **15**, 503], are now treated in full detail. The method yields inter alia a new proof of results of Mal'cev [Izv. Akad. Nauk SSSR. Ser. Mat. **13** (1949), 201–212; Dokl. Akad. Nauk SSSR (N.S.) **62** (1948), 745–748; MR **10**, 507, 232] on locally nilpotent locally infinite groups, and incidentally weakens the assumption of local nilpotence to nilpotence of all 3-generator subgroups. One remarkable theorem establishes a correspondence between Lie rings of prime characteristic p with the property that every 3-generator subring has class less than p, and p-groups in which every 3-generator subgroup has class less than p; in this correspondence subrings and subgroups correspond to each other, and in particular ideals and normal subgroups. An illustrative example is made of a Lie ring of characteristic 5 and the corresponding group; this is of exponent 5 and order 5^{91}, has 4 generators and class 5, and every 3-generator subgroup of it has class 4 or less. For numerous further results the reader has to be referred to the paper itself.

B. H. Neumann (Zbl **55** (1955), 251).

Sands, Arthur D. On the factorisation of finite abelian groups. Acta Math. Acad. Sci. Hungar. **8** (1957), 65–86.

If a finite abelian group G has two subsets A and B such that each g in G has a unique representation as ab where $a\in A$ and $b\in B$, then $AB=G$ is called a factorisation of G. If there exists a non-trivial $g\in G$ such that $gA=A$, then A is called periodic. If, for all factorisations AB of G, at least one of A and B is periodic, then G is called good. The author shows that if G is finite cyclic and if the number of elements of A is a prime power, then one of A and B is periodic. He settles the question as to which finite cyclic groups are good, extending results of de Bruijn [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 258–264; MR **15**, 8]. Cyclotomic polynomials are used throughout in the proofs. He corrects a result of Hajós [Časopis Pěst. Mat. Fys. **74** (1949), 157–162; MR **13**, 623] which leads to all the factorisations of a good group.

F. Haimo (St. Louis, Mo.).

Maranda, J.-M. On the equivalence of representations of finite groups by groups of automorphisms of modules over Dedekind rings. Canad. J. Math. **7** (1955), 516–526.

Es sei g ein Dedekindscher Ring (der Charakteritik 0), G eine endliche Gruppe. Eine g-Darstellung von G ist eine Darstellung durch Automorphismen eines g-Moduls M, welcher torsionfrei und über g endlich erzeugt ist; M ist also ein Modul über dem Gruppenring von G bezüglich g. Ferner bezeichne K den Quotientenkörper von g; es wird nun eine volle Klasse C von g-Darstellungen der Gruppe G betrachtet, die einander K-äquivalent sind (dies bedeutet, dass die zugehörigen auf K erweiterten Moduln KM einander isomorph sind über dem Gruppenring von G bezüglich K), und es wird untersucht, wie diese Klasse C in Klassen unter sich g-äquivalenter Darstellungen zerfällt; r sei die Anzahl dieser Klassen. Für ein Primideal $\mathfrak{p}$ von g sei $g_\mathfrak{p}$ der Ring aller $\mathfrak{p}$-regulären Elemente von K;

die Klasse C zerfällt in r_p Klassen g_p-äquivalenter Darstellungen. Schliesslich sei $\bar{r}$ die Anzahl der Geschlechter in welche C zerfällt, wobei zwei Darstellungen zum selben Geschlecht gerechnet werden, falls sie für alle Primideale p g_p-äquivalent sind. Es gilt dann $\bar{r}=\prod r_p$, das Produkt erstreckt über alle Primideale p, die in der Gruppenordnung aufgehen. Wenn jede Darstellung aus C absolut irreduzibel ist (d.h., wenn alle KM absolut irreduzibel sind über dem Gruppenring von G bezüglich K), so gilt überdies $r=h\cdot\bar{r}$, wo h die Idealklassenzahl von g ist.

Im Falle von Darstellungen von G im Integritätsbereich g der ganzalgebraischen Zahlen eines algebraischen Zahlkörpers K ist h endlich, und auch alle r sind endlich [Maranda, Canad. J. Math. **5** (1953), 344–355; MR **15**, 100], so dass also die Endlichkeit von r resultiert [vgl. Zassenhaus, Abh. Seminar Hamburg **12** (1938), 276–288].

B. Eckmann (Zürich).

Mal'cev, A. I. Remark on partially ordered groups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 3–5. (Russian)

Let G be a partially ordered group. If every subgroup H with finitely many generators has a central system of convex subgroups in H, then G has a central system of convex subgroups. If every subgroup with finitely many generators has a solvable system of convex normal subgroups of G, then G has a solvable system of convex normal subgroups. *R. A. Good* (College Park, Md.).

Heine, V. Irreducible representations of the full Lorentz group. Phys. Rev. (2) **107** (1957), 620–623.

This paper gives an account of the extension of the representation theory of the proper Lorentz group to that of the full Lorentz group. Though the ideas are old it is worth having them set forth so clearly and compactly.

G. de B. Robinson (Toronto).

Hion, Ya. V. Ordered semigroups. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 209–222. (Russian)

Let P be a semigroup (non-void set with a binary associative multiplication), completely ordered by an order relation $\leqq$ such that: $\alpha, \beta, \gamma \in P$ and $\alpha \geqq \beta$ imply $\alpha\gamma \geqq \beta\gamma$ and $\gamma\alpha \geqq \gamma\beta$; P contains an element 0 such that $0\alpha=\alpha 0=0$ for all $\alpha \in P$; $0 \leqq \alpha$ for all $\alpha \in P$; $\alpha\gamma=\beta\gamma \neq 0$ or $\gamma\alpha=\gamma\beta \neq 0$ imply $\alpha=\beta$, for all $\alpha, \beta, \gamma \in P$. Then P is called an ordered semigroup. Let $K=\{\alpha: \alpha \in P, \alpha^2 \leqq \alpha\}$; elements of K are called integral. If $K=P$, then P is called an integral ordered semigroup. Let $M=\{\alpha: \alpha \in P, \alpha=0 \text{ or } \alpha^2<\alpha\}$. A subset A of P is called convex if $\alpha, \beta \in A$, $\gamma \in P$, and $\alpha<\gamma<\beta$ imply $\gamma \in A$. Sample theorems follow. The set K is a convex subsemigroup of P. The set $P \cap K'$ is a convex subsemigroup (without zero) of P. Every convex subset of K that contains 0 is a 2-sided ideal in K. The set K contains at most one element ε that is not in M; clearly $\varepsilon^2=\varepsilon$. The set M contains all proper convex left (right) ideals in P. If $M \neq P$, then P contains a maximal convex 2-sided ideal Λ.

The set N of all elements $\alpha \in P$ such that $\alpha^n=0$ for some positive integer n is called the radical of P. The set N is a convex 2-sided ideal in P. The inclusions $\{0\} C N C \Lambda C M C P$ (Λ appears only when it exists) always hold. The quotient semigroup P/N contains no zero-divisors; P/Λ contains no proper 2-sided convex ideals. {The example beginning on line −3, p. 213, is invalid unless a zero is added.}

The author next takes up the Archimedean property for pairs of elements of P, in the sense of N. G. Alimov [same Izv. **14** (1950), 569–576; MR **12**, 480], showing that P can be partitioned into pairwise disjoint sets $\tilde{\alpha}$, where $\tilde{\alpha}$ consists of all $\beta \in P$ such that α and β are mutually Archimedean. These classes form a semigroup under the operation $\tilde{\alpha}\tilde{\beta}=\widetilde{\alpha\beta}$; for $\alpha, \beta \in K$, $\tilde{\alpha}\tilde{\beta}=\min(\tilde{\alpha}, \tilde{\beta})$, while for $\alpha, \beta \in P \cap K'$, $\tilde{\alpha}\tilde{\beta}=\max(\tilde{\alpha}, \tilde{\beta})$.

The final section deals with integral ordered semigroups. The proof of the third assertion of Theorem 9 is invalid (the element σ mentioned on line −7 of p. 216 need not exist). The notion of anomalous pairs of elements in the sense of Alimov is introduced. A pair α, β of elements of the integral ordered semigroup P is said to be anomalous ("near" in the terminology of the present paper) if $\alpha^{n+1} \leqq \beta^n \leqq \alpha^n$ for all positive integers n. It is claimed that the relation of anomaly is reflexive, symmetric, and transitive, and the semigroup P is partitioned on this assumption. However, α, β and β, α both anomalous implies $\alpha=\beta$, and therefore the results of the last section appear to be in doubt. *E. Hewitt* (Seattle, Wash.).

Hion, Ya. V. Rings normed by the aid of semigroups. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 311–328. (Russian)

Let R be an associative ring, and P an ordered semigroup in the sense of the preceding review. (Notation and terminology are borrowed where needed from this review.) R is said to be normed by the aid of P if there is a single-valued mapping w of R onto P such that: $w(a-b) \leqq \max(w(a), w(b))$ and $w(ab)=w(a)w(b)$ for all $a, b \in R$; $w(a)=0$ only if $a=0$. The author first gives a general, and fairly obvious, necessary and sufficient condition for a ring R to admit such a norm, in terms of subgroups of the additive group of R. Every ordered semigroup P acts as the norm for some ring. (The semigroup algebra of P over any integral domain, modulo the ideal of multiples of 0, will do.) A subgroup G of R, where R is normed, is said to be convex if $b \in G$, $a \in R$, and $w(a) \leqq w(b)$ imply that $a \in G$. Normed homomorphic images of a normed ring R are identified in a natural way as those homomorphs of R whose kernels are convex. If R is normed, $R^2 \neq \{0\}$, and R is a direct sum of subrings A and B, then either A or B is a zero ring. The set $\mathcal{N}$ of all nilpotent elements of R is equal to the Baer, Köthe, and Levitski radicals of R. $R/\mathcal{N}$ has no 0-divisors. The author also studies completely ordered rings; such rings can always be normed by the aid of the additive Archimedean classes of their non-negative elements. *E. Hewitt.*

Gourevitch, Georges. Sur certains ensembles à loi de composition aléatoire. C. R. Acad. Sci. Paris **245** (1957), 614–617.

Gourevitch, Georges. Sur l'étude de la condition d'associativité des ensembles à loi de composition aléatoire. C. R. Acad. Sci. Paris **245** (1957), 778–780.

In the first paper, the author studies the groupoid E generated by sequences $\{\pi_a\}$ of n numbers whose elements $\pi_a{}^k$ satisfy $0 \leqq \pi_a{}^k \leqq 1$, $\sum_k \pi_a{}^k=1$ ($k=1, \cdots, n$). Multiplication is defined by $(\pi_a \cdot \pi_b)^k=\pi_a{}^s \pi_b{}^t p_{st}{}^k$ (summation rule) with $0 \leqq p_{st}{}^k \leqq 1$, $\sum_k p_{st}{}^k=1$. An isomorphism is exhibited between E and each of two conjugate classes of matrices. Associativity of E is related to commutativity of conjugate matrices. An example of results given without proof is the following. If $p_{st}{}^k$ is chosen so that E is associative, and P_t and $P_t{}^*$ are the

conjugate matrices corresponding to the element of E having 1 for its ith entry then E is commutative if and only if $P_i=P_i^*$.

In the second paper, conditions are given under which E is associative in terms of special properties of the spectra of the associated conjugate systems of matrices. *H. P. Kramer* (Murray Hill, N.J.).

See also: Algebra: Séminaire Dubreil et Pisot. **Combinatorial Analysis:** Herrera. **Algebras:** Higman.

THEORY OF NUMBERS

General Theory of Numbers

Čulík, Karel. Über eine Eigenschaft der ganzzahligen nichtnegativen Lösungen der Gleichung $\sum_{i=1}^{k} r_i=n$. Časopis Pěst. Mat. **82** (1957), 353–359. (Czech. Russian and German summaries)

Let n and k be natural numbers, and $r_1, r_2, \cdots, r_k$ non-negative integers satisfying $\sum_{i=1}^{k} r_i=n$ ($r_1\leqq r_2\leqq\cdots\leqq r_k$). The author has succeeded in picking out from among all such partitions of n into k parts the partition which minimises, for every integer $c\geqq 0$, the expression $\sum_{i=1}^{k}\binom{r_i}{c}$.

He was led to this investigation by his own work, and the work of others, on a problem of K. Zarankiewicz [Kövari, Sós, and Turán, Colloq. Math. **3** (1954), 50–57; MR **16**, 456; K. Čulík, Acta Acad. Sci. Čechoslovenicae Basis Brunensis **27** (1955), 341–348; Ann. Polon. Math. **3** (1956), 165–168; MR **18**, 792, 459] concerning matrices with elements equal to 0 or 1, and he describes the application of his new result. He also describes how this result leads him to a conjecture on a problem of Turán [Elem. Math. **11** (1956), 40]. *H. Halberstam.*

McCarthy, Paul J. Remarks concerning the non-existence of odd perfect numbers. Amer. Math. Monthly **64** (1957), 257–258.

The author obtains new necessary conditions for odd perfect numbers. It is well known that an odd perfect number N has the form $N=p^{\alpha}q_1^{2\beta_1}\cdots q_t^{2\beta_t}$. The author proves that $N\not\equiv 2 \pmod 3$. For $N\equiv 1 \pmod 3$ he proves that $\alpha+1\not\equiv 0 \pmod 3$. More generally, let r be any prime not dividing N and let k be the exponent to which p belongs mod r; then N is not perfect if $\alpha+1\equiv 0 \pmod{kr}$. {It must be remarked that this condition can easily be improved and generalized. N is not perfect if (1) $\alpha+1\equiv 0 \pmod r$ for $p\equiv 1 \pmod r$ or (2) $\alpha+1\equiv 0 \pmod k$ for $p\not\equiv 1 \pmod r$. If (1) holds, $\sigma(p^{\alpha})\equiv\alpha+1\equiv 0 \pmod r$ and if (2) holds, $\sigma(p^{\alpha})=(p^{\alpha+1}-1)/(p-1)\equiv 0 \pmod r$, since $1\equiv p^k\equiv p^{\alpha+1} \pmod r$. Hence, r would be a divisor of $\sigma(p^{\alpha})$, and of N. Similar results can be obtained for each q_r.} Moreover, the author improves results of Steuerwald [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1937**, 68–72] and Kanold [Ber. Math.-Tagung Tübingen **1946**, 84–87; MR **9**, 78]. *A. Brauer* (Chapel Hill, N.C.).

McCarthy, Paul J. Note on the distribution of the totatives. Amer. Math. Monthly **64** (1957), 585–586.

This note is based on a paper by the reviewer [Canad. J. Math. **7** (1955), 347–357; MR **16**, 998] concerned with the number of numbers prime to n lying in each of the k subintervals $nq/k<m<n(q+1)/k$ ($q=0(1)k-1$, $k<n$). The case of uniform distribution, when each of these k intervals contains the same number of these "totatives", is especially interesting. This situation occurs whenever n is divisible by k^2 or by a prime of the form $kx+1$. The author shows that for k a prime uniform distribution occurs for no other kinds of n. For k composite there may be other cases of uniform distribution. Such cases always exist when k is divisible by a square. The square-free case presents difficulty. *D. H. Lehmer.*

Moessner, Alfred. Einige Bemerkungen zu dem magischen Quadrat mit 3 Feldern. Euclides, Madrid **17** (1957), 6–8.

Xiroudakis, Georges. Les systèmes multigrades dont les termes sont formés au moyen des nombres de Pythagore. Mathesis **66** (1957), 7–16.

A solution in integers of $A^2=B^2+C^2$ provides a solution of the diophantine system

$$A,\ A,\ 2A\ \overset{3}{-}\ A+B,\ A-B,\ A+C,\ A-C.$$

In this paper the author discusses the solutions of the above and similar systems of diophantine equations obtained from solutions of $A^2=B^2+C^2$. Numerous numerical examples are given. *W. H. Simons.*

See also: Algebra: Séminaire Dubreil et Pisot. **Elementary Algebra:** Novák. **Fields, Rings:** Lewis. **Analytic Theory of Numbers:** Hooley.

Analytic Theory of Numbers

★ **Meyer, Curt. Die Berechnung der Klassenzahl Abelscher Körper über quadratischen Zahlkörpern.** Akademie-Verlag, Berlin, 1957. ix+132 pp. DM 29.00.

The author gives complete proofs of analytic formulas for the product of regulator and class number of subfields of an abelian extension of a quadratic extension of the rational field. These results have already been reported by H. Hasse [Rend. Mat. e Appl. (5) **10** (1951), 84–95; MR **14**, 140]. The structure of the formulas is summarized in the review cited. *G. Whaples.*

Selberg, A. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.

This article outlines an extensive theory concerning certain eigenvalue problems and Dirichlet series associated with a group G of isometries of a Riemannian space S. The metric $ds^2=\sum g_{ij}dx^i dx^j$ of S is to be positive definite, the g_{ij} being analytic in the $x^1, \cdots, x^n$. G is to be locally compact and must act transitively on S, but need not be the full group of isometries. The eigenvalue problems relate to linear operators L on functions $f(x)$ defined for $x\in S$. Such an L is termed invariant if it commutes with the operation $f(x)\to f(mx)$ for all $m\in G$. An integral L of the form $Lf(x)=\int_S k(x,y)f(y)dy$, where dy is the invariant volume element derived from the metric, is invariant if k is a "point-pair invariant", i.e., if $k(mx, my)=k(x, y)$ for all $x, y\in S$, $m\in G$. Invariant differential operators are constructed first at some $x_0\in S$, at which point they are equivalent to a ring of polynomials in $\partial/\partial x^1, \cdots,$

$\partial/\partial x^n$ with constant coefficients, subject to invariance under a certain group of orthogonal transformations; they can then be extended to the whole of S by the rule $[Lf(x)]_{x=mx_0}=[Lf(mx)]_{x=x_0}$. For this ring there exists at least one minimal basis $D_1, \cdots, D_l$. Since further developments depend on the commutativity of the invariant operators, a sufficient condition to ensure this is postulated, namely that S is "weakly symmetric". There is to be a fixed isometry μ of S, such that $\mu G\mu^{-1}=G$, $\mu^2 \in G$, with the property that for any $x, y \in S$ there is an $m \in G$ such that $mx=\mu y$ and $my=\mu x$. This requirement is in fact weaker than that of symmetry in the sense of E. Cartan.

An eigenfunction $f(x)$ is to be a simultaneous solution of $D_i f(x)=\lambda_i f(x)$ $(i=1, \cdots, l)$ for some set of constants λ_i; when suitably normed and symmetrised it is uniquely fixed by the λ_i. It is necessarily also an eigenfunction of any other invariant L, so that $Lf(x)=\Lambda(\lambda_1, \cdots, \lambda_l)f(x)$. Eigenfunctions can be given explicitly if one assumes there to exist a subgroup T of G which is simply transitive on S, and also functions $\phi(t)$ on T such that $\phi(t_1t_2)=\phi(t_1)\phi(t_2)$.

The first section of the article concludes by illustrating the concepts so far introduced by the case in which S is the space of positive definite symmetric n by n matrices $Y=(y_{ij})$ with the metric $ds^2=\sigma(Y^{-1}dY\,Y^{-1}dY)$, σ=trace, and the isometries G are $Y \to AYA'$, where A is any real non-singular n by n matrix, A' being its transpose; μ is taken as $Y \to Y^{-1}$. For this case the author sketches the mode of calculation of the eigenvalues of integral operators, making contact with an investigation of H. Maass [same J. **19** (1955), 1–23; MR **17**, 588].

The next stage is to consider a discrete subgroup Γ of G, whose action on S is properly discrete; let $\mathfrak{D}$ be a fundamental domain of Γ in S. Let $\chi(M)$, $M \in \Gamma$, be a representation of Γ by ν by ν matrices, and consider ν-vectors or column matrices $F(x)$, $x \in S$, such that $F(Mx)=\chi(M)f(x)$. The formula

$$\int_S k(x, y)F(y)dy = \int_{\mathfrak{D}} K(x, y; \chi)F(y)dy$$

where $K(x, y; \chi)=\sum_{M\in\Gamma}\chi(M)k(x, My)$ leads to a study of the latter integral operator in the Hilbert space of $F(x)$ with inner product $(F_1, F_2)=\int_{\mathfrak{D}}\overline{F}_1'(x)F_2(x)dx$, where $\overline{F}_1'$ is the conjugate transpose of $\overline{F}_1$.

Under certain conditions, including the compactness of $\mathfrak{D}$, one has an eigenfunction expansion of $K(x, y; \chi)$ of the form $\sum_{M\in\Gamma}\chi(M)k(x, My)=\sum_t h(\lambda^t)F_t(x)\overline{F}_t'(y)$, where $\lambda^t=\lambda_1^t, \cdots, \lambda_l^t$, and the F_t are eigenfunctions of the $D_1, \cdots, D_l$. Taking the trace of this identity, and transforming the right-hand side into a sum over conjugacy classes in Γ, the author obtains the "trace-formula" ((2.12), p. 67), constituting the centre of his investigation; it may be used to investigate the distribution of either the λ^t or the conjugacy classes in Γ, and may also be viewed as a generalisation of the Poisson summation formula. The section concludes with a brief account of generalisations, one in connection with the Hecke operator for the modular group, and the other concerning the case in which $\mathfrak{D}$ has finite volume but is not compact, when the spectrum may cease to be discrete.

The remainder of the paper deals with special cases. There is first the case when S is the hyperbolic plane $z=x+iy$, $y>0$, with the metric $ds^2=y^{-2}(dx^2+dy^2)$, and the isometries of G are $z \to (az+b)/(cz+d)$, $ad-bc=1$, a, b, c and d real. The eigenfunctions have the form y^s, the point-pair invariants the form $k(z, z')=k(|z-z'|^2/(yy'))$; formulae concerning the eigenvalues are also given.

The trace-formula is given in full for the case in which Γ has a compact $\mathfrak{D}$, and leads to the consideration of the zeta-function

$$Z_\Gamma(s; \chi)=\prod_{\{P\}_\Gamma}\prod_{k=0}^{\infty}|E_\nu-\chi(P)(N\{P\})^{-s-k}|,$$

for $\Re\{s\}>1$, where E_ν is the ν by ν identity matrix, $|\cdot|$ indicates the determinant, the first product is taken over a set of representatives, one from each primitive hyperbolic conjugacy class $\{P\}_\Gamma$ of Γ, and the norm $N\{P\}$ is derived from that element of the conjugacy class which is a pure magnification. $Z_\Gamma(s; \chi)$ and $Z_\Gamma(1-s; \chi)$ are connected by a functional equation. The complex zeros have real part $\frac{1}{2}$. After discussing modifications required when $\mathfrak{D}$ is not compact, the author discusses other special spaces S. Some details are given for the case in which S is the space of 3 by 3 symmetric matrices with determinant unity, Γ is the group of 3 by 3 matrices with determinant unity and integral rational elements; there arises the Dirichlet series in two variables

$$\sum (X'YX)^{-s}(Z'Y^{-1}Z)^{-s'}=\zeta_\Gamma(s,s'),$$

where the summation is over pairs of column vectors X, Z with integral rational components subject to $X'X>0$, $Z'Z>0$, $X'Z=0$. This series satisfies four functional equations. Generalisations are indicated.

The final section of the paper deals with the space (z, ϕ), $z=x+iy$, $y>0$, ϕ being identified with $\phi+2\pi$. The group G has elements m_α, where

$$m_\alpha(z, \phi)=((az+b)/(cz+d), \phi+\arg(cz+d)+\alpha),$$

where a, b, c and d are as before; a suitable metric is $ds^2=y^{-2}(dx^2+dy^2)+(d\phi-\frac{1}{2}y^{-1}dx)^2$. The author outlines the argument leading to formulae for the number of analytic automorphic forms of various classes. The special case of one-dimensional $\chi(M)$, together with the generalisation leading to Hecke operators, yields a formula for the trace of this operator acting on the space of cusp-forms of dimension $-k$, a special case being an explicit formula for Ramanujan's function $\tau(n)$. Extensions are indicated to spaces of higher dimensionality.

F. V. Atkinson (Canberra).

Hooley, C. On the representation of a number as the sum of two squares and a prime. Acta Math. **97** (1957), 189–210.

On the basis of the so-called "extended Riemann hypothesis" the author proves an asymptotic formula, conjectured by Hardy and Littlewood, for the number of representations of a large number as a sum of two squares and a prime. On the same assumption he proves that there are infinitely many primes of the form x^2+y^2+1 (a well-known conjecture which had not, so far, been connected with the extended Riemann hypothesis). The Hardy-Littlewood conjecture for the case of 4 squares and a prime was proved (however, without any hypothesis) by the reviewer [Acta Arith. **1** (1935), 115–122] and for a greater number of squares and primes by A. Walfisz [ibid. **1** (1935), 123–160]; Estermann [Proc. London Math. Soc. (2) **42** (1937), 501–516] and Halberstam [ibid. **53** (1951), 363–380; MR **13**, 112] dealt with other cases of r squares and s primes. The author's results are deep, and difficult arguments are necessary to achieve the proofs.

S. Chowla (Princeton, N.J.).

Arkhangelskaya, V. M. Some calculations connected with Goldbach's problem. Ukrain. Mat. Ž. **9** (1957), 20–29. (Russian. English summary)

Vinogradov proved that every sufficiently large odd integer n ($n>n_0$) can be written as the sum of three odd primes [Trudy Mat. Inst. Steklov. **10** (1937)]. In 1939 K. G. Borozdkin calculated the constant n_0 to be equal to $10^{10^{10^{17.86}}}$ [see Čudakov, Introduction to the theory of Dirichlet's L-functions, OGIZ, Moscow-Leningrad, 1947, p. 201; MR **11**, 234]. The Goldbach conjecture has been verified for integers $n\leqq 10^7$ by using a table of primes. In the present paper the author discusses the distribution of "Vinogradov" numbers (i.e., numbers representable as a sum of three odd primes) in the interval $10^7<n<n_0$. He proves: (1) for each $x>x_0=10^{10^{8.324}}$ the interval $[x, x+(\ln x)^{14}]$ contains at least one Vinogradov number; (2) assuming the truth of the Riemann hypothesis, for each $x>x_0=10^{30}$ the interval $[x, x+(\ln x)^{6.065}]$ contains at least one even integer which is representable as the sum of two odd primes. *W. H. Simons* (Vancouver, B.C.).

Cassels, J. W. S. Über Basen der natürlichen Zahlenreihe. Abh. Math. Sem. Univ. Hamburg **21** (1957), 247–257.

A sequence $a_1<a_2<\cdots$ of integers is called a basis of order k of the integers, if every sufficiently large integer is the sum of k or fewer a's. The author constructs for every k a basis of order k satisfying

$$a_n=c_k n^k+O(n^{k-1}). \tag{1}$$

The question of the smallest possible value of c_k is left open (it probably is quite difficult). He constructs a sequence with $c_2=1/27$. The question whether in (1) $O(n^{k-1})$ can be replaced by $o(n^{k-1})$ is also left open.

P. Erdős (Toronto, Ont.).

Wang, Yuan. On sieve methods and some of their applications. Sci. Record (N.S.) **1** (1957), no. 3, 1–5.

The method of P. Kuhn [Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 160–168; MR **16**, 676] is further developed to give improved results as follows. For sufficiently large x there exists a number n in the interval $x-x^{\delta_i}<n\leqq x$ with at most i prime factors, where $\delta_2=10/17$, $\delta_3=20/49$, $\delta_6=1/5$, $\delta_{103}=3/301$. For infinitely many integers x, x^3+2 has at most 4 prime factors. *D. H. Lehmer* (Berkeley, Calif.).

See also: Algebraic Geometry: Lang.

Theory of Algebraic Numbers

Billevič, K. K. On units of algebraic fields of third and fourth degree. Mat. Sb. N.S. **40(82)** (1956), 123–136. (Russian)

The author presents methods for finding systems of fundamental units for algebraic number fields of the third and fourth degrees. These methods seem to be suitable for the actual construction of tables of fundamental units for such fields and apparently may be generalized to the case of fields of arbitrary degree. (The author asserts that the method given by Voronoĭ in his doctoral dissertation [Warsaw, 1896; see Collected Works in three volumes, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1952, 1953; MR **16**, 2] does not have these advantages.) The paper contains the following tables: (i) systems of fundamental units for totally real cubic fields having discriminants less than 1297, (ii) systems of fundamental units for totally real quartic fields having discriminants less than 7169, (iii) systems of fundamental units for partially real quartic fields having discriminants numerically less than 849. *P. T. Bateman* (Urbana, Ill.).

Jehne, Wolfram. Zur modernen Klassenkörpertheorie. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1954**, no. 3, 8 pp. (1954).

Let K be a Galois extension of an algebraic number field Ω, with Galois group $\mathfrak{g}$. Let $G_{K/\Omega}$ be Weil's [J. Math. Soc. Japan **3** (1951), 1–35; MR **13**, 439] group for K/Ω, which is an extension of the idèle-class group C_K of K by $\mathfrak{g}$ with a certain canonical factor set, homomorphically mapped on the Galois group $\mathfrak{G}_{K/\Omega}$ over Ω of the maximal abelian extension of K, and possesses certain properties in connection with this homomorphic map, transfer map, Artin-Chevalley symbol and intermediate fields. It is asserted first that the algebraic commutator group $G'_{K/\Omega}$ of $G_{K/\Omega}$ coincides with the topological one, so that $G_{K/\Omega}/G'_{K/\Omega}$ is isomorphic to C_Ω by the transfer map, refining one of the said properties of $G_{K/\Omega}$ in Weil, loc. cit. Then the algebraic and topological commutator groups of $\mathfrak{G}_{K/\Omega}$ also coincide. (For these cf. also a paper by Witt [Rend. Mat. e Appl. (5) **14** (1954), 125–129; MR **16**, 900] following the present one.) For a fixed K/Ω the groups $G_{\Lambda/\Omega}$ with intermediate Galois extensions Λ/Ω are uniquely determined by the said properties. The homomorphism of $G_{K/\Omega}$ onto $\mathfrak{G}_{K/\Omega}$ is characterized uniquely in that it induces for every place of K the canonical isomorphism of the local Weil group and Frobenius group. Further, the factor group $N_{K/\Omega}$ of the multiplicative group of Ω modulo the elements which are everywhere norm for K/Ω is seen to be finite and to have exponent dividing $[K:\Omega]/(\text{l.c.m. of local degrees})$. The order of $N_{K/\Omega}$ is reduced to three factors, and a system of generators of $N_{K/\Omega}$ is obtained for abelian K/Ω in terms of Teichmüller's 3-cocycle. The result seems to lie in close relationship to recent results of Artin-Tate. A detailed exposition is promised elsewhere.

T. Nakayama (Zbl **58** (1957), 29).

See also: Groups and Generalizations: Kantz. **Analytic Theory of Numbers:** Meyer. **Algebraic Geometry:** Lang.

Geometry of Numbers

Lekkerkerker, Cornelis Gerrit. Una questione di approssimazione diofantea e una proprietà caratteristica dei numeri quadratici. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 257–262.

Completes the proofs of results already enunciated in the first part [same Rend. (8) **21** (1956), 179–185; MR **19**, 124]. *J. W. S. Cassels* (Cambridge, England).

Rankin, R. A. Diophantine approximation and horocyclic groups. Canad. J. Math. **9** (1957), 277–290.

Let Γ be a real zonal horocyclic group of the first kind (i.e., a Fuchsian group of the first kind with a finite number of generators and real axis as principal circle, and with ∞ as a parabolic fixed point), and let $Tz=(az+b)/(cz+d)$ $(ad-bc=1)$, a, b, c, d real. The principal problem considered is the diophantine approximation problem of determining h so that the inequality (*) $|\omega-a/c|=|\omega-T\infty|<1/hc^2$ will have infinitely many solutions $a/c=T\infty$, where $T\in\Gamma$ and ω is not a parabolic fixed point for Γ. If T denotes the coefficient matrix of Tz, if U^λ denotes the matrix for $U^\lambda z=z+\lambda$, where λ is the cusp width of Γ at

∞, and if Γ_U denotes the cyclic subgroup of Γ generated by $U^\lambda z$, then an equivalent statement of the problem is the determination of h so that there will be infinitely many left cosets of Γ_U in Γ whose members T satisfy (*). A lower bound, h_Γ, and two upper bounds, h_Γ' and h_Γ'', are determined for $h(E, \Gamma) = \sup h$ in the set of all h for which (*) holds for transformations T belonging to infinitely many left cosets of Γ_U in Γ, and it is shown that $h = h_\Gamma$ is a solution of the problem. A lower bound for $h(E, \Gamma)$, and also a solution of the problem, had previously been obtained by Lehner [Pacific J. Math. **2** (1952), 327–333; MR **14**, 359]. For the corresponding problem of approximating ω by $ST\infty$, where S is an arbitrary but fixed transformation having real axis as fixed circle, it is shown that $h(S, \Gamma) = h(E, \Gamma)$. For the modular group $\Gamma(1)$ and for ω irrational the inequality $\sqrt{3} \leqq h(E, \Gamma(1)) \leqq \sqrt{5}$ is obtained. Ford [Proc. Edinburgh Math. Soc. **35** (1917), 59–65], in a detailed study that was restricted to $\Gamma(1)$, showed that $h = \sqrt{5}$ is the maximal solution of the corresponding problem. It is remarkable that, for the principal congruence group $\Gamma(2)$ of level 2, the present general treatment yields $h = h(S, \Gamma(2)) = 1$, $S \in \Gamma(1)$, which had been shown by the reviewer [Bull. Amer. Math. Soc. **46** (1940), 124–129; MR **1**, 203] to be the maximal solution of the corresponding diophantine approximation problem. The proofs depend on Hermite's geometric formulation of the problem and Ford's determination of fundamental regions by the method of isometric circles.

W. T. Scott (Evanston, Ill.).

See also: Algebra: Séminaire Dubreil et Pisot. **Theory of Algebraic Numbers:** Billevič.

ANALYSIS

Functions of Real Variables

Efimov, A. V. Estimation of the modulus of continuity of functions of class $\tilde{H}_2^1$. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 283–288. (Russian)

Let $\tilde{H}_2^1$ be the class of all continuous 2π-periodic functions such that for all x and $h > 0$, (*) $|f(x+h) - 2f(x) + f(x-h)| \leqq h$. Zygmund [Duke Math. J. **12** (1945), 47–76; MR **7**, 60] proved that $\omega(h) = O(h \log 1/h)$, where $\omega(h) = \sup \omega(f, h)$ and $\omega(f, h)$ is the modulus of continuity of f, the supremum being taken for all $f \in \tilde{H}_2^1$. The author finds the asymptotic formula

$$\omega(h) = [2 \log(2^1 + 1)]^{-1} h \log \frac{1}{h} + O(h);$$

he uses methods of Timan [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 243–254; MR **13**, 17], who solved this problem for functions satisfying (*), defined on a closed interval $[a, b]$ and vanishing at a and b. *G. G. Lorentz.*

Marcus, Solomon. Un critère de finitude pour les fonctions sous-additives. C. R. Acad. Sci. Paris **244** (1957), 2221–2222.

Let the real function $f(x)$ be subadditive on the interval $(0, \infty)$; that is, let $f(x)$ be such that $f(x+y) \leqq f(x) + f(y)$ for x and y on this interval. Let E be a set of points on $(0, \infty)$ such that for all $n \geqq N$ the set of points $(0, 1/n) \cap S(E)$ contains an interval, where N is some fixed integer and $S(E)$ consists of the points $x+y$ with $x \in E$ and $y \in E$. The author shows that if $f(x) < \infty$ for $x \in E$ then $f(x) < \infty$ for $x \in (0, \infty)$, and that if, in addition, there is a point x_0 such that $f(x_0) > -\infty$ then $f(x)$ is finite on $(0, \infty)$.

E. F. Beckenbach (Los Angeles, Calif.).

Marcus, Solomon. Critères de majoration pour les fonctions sous-additives, convexes ou internes. C. R. Acad. Sci. Paris **244** (1957), 2270–2272.

With the notation of the preceding review, it is shown that if the function $f(x)$, subadditive on $(0, \infty)$, is bounded from above on a set of points E on $(0, \infty)$ such that for all $n \geqq N$ the set of points $(0, 1/n) \cap S(E)$ contains an interval, then $f(x)$ is bounded from above on every compact interval on $(0, \infty)$, and that if, in addition, there is a point x_0 such that $f(x_0) > -\infty$ then $f(x)$ is bounded on every compact interval on $(0, \infty)$. Next, it is shown that if a function $f(x)$, convex on an interval (a, b), is bounded from above on a set of points E of (a, b) such that, for some n, $S^n(E)$ contains an interval, then $f(x)$ is continuous on (a, b). [Cf. S. Kurepa, Glasnik Mat. -Fiz. Astr. Ser. II. **11** (1956), 89–94; MR **19**, 408.]

A function $f(x)$ is said to be weakly interior if, for all x, y in the interval of definition, we have $\min[f(x), f(y)] \leqq f(\frac{1}{2}(x+y)) \leqq \max[f(x), f(y)]$. It is shown that if such a function is bounded from above (or below) on a subinterval of (a, b) then it is bounded from above (or below), with the same bound, throughout (a, b). *E. F. Beckenbach.*

Diananda, P. H. On Taylor's theorem with remainder. Amer. Math. Monthly **64** (1957), 492–495.

The author gives a proof of Taylor's theorem with remainder of the form

$$\frac{h^n (1-\theta)^\nu f^{(n)}(a+\theta h)}{(n-1)!(n-\nu)},$$

for $\nu < n$, on the assumption that $f(x)$ is continuous in $a \leqq x \leqq a+h$, $f^{(n-1)}(x)$ is continuous in $a \leqq x < a+h$, and $f^{(n)}(x)$ exists in $a < x < b$. References are given to the literature relative to proofs under the foregoing weak hypothesis concerning $f^{(n-1)}(x)$. A similar result has been given by Grimshaw [Proc. Cambridge Philos. Soc. **52** (1956), 376–378; MR **17**, 830]. *E. F. Beckenbach.*

Weinstock, Robert. On continuous differentiability. Amer. Math. Monthly **64** (1957), 492.

Suppose the derivative, $f'(x)$, of a function $f(x)$ exists on an open interval $I \supset [a, b]$. The author proves that $f'(x)$ is continuous on $[a, b]$ if and only if, given any $\varepsilon > 0$, there exists a $\delta > 0$ independent of x such that $|h^{-1}[f(x+h) - f(x)] - f'(x)| < \varepsilon$ whenever $0 < |h| < \delta$, $x \in [a, b]$, $x+h \in I$.

Pandres, Dave, Jr. On higher ordered differentiation. Amer. Math. Monthly **64** (1957), 566–572.

Giuliano, Landolino. Sul limite di integrali del tipo $\int_a^b f(x)\phi(hx, \tau)dx$, quando $h \to +\infty$. Boll. Un. Mat. Ital. (3) **12** (1957), 186–191.

The author gives two proofs (one direct, one a specialization from the classical general theory of singular integrals) of the formula

$$\lim_{n \to \infty} \int_a^b f(x)\phi(nx, \tau)dx = T^{-1} \int_0^1 \phi(t, \tau)dt \int_a^b f(x)dx,$$

where ϕ is periodic of period T in its first argument and f and ϕ are subject to suitable auxiliary restrictions.

R. P. Boas, Jr. (Evanston, Ill.).

Katznelson, Yitzhak. Sur la décomposition des classes de fonctions. C. R. Acad. Sci. Paris **244** (1957), 2877–2879.

The author proves a lemma on topological Abelian groups which contains the essential part of a lemma used by Markouchevitch [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **44** (1944), 262–264; MR **6**, 179] to establish the decomposition of a continuous function into a sum of Bernstein quasi-analytic functions. He uses this to establish decomposition theorems of Mandelbrojt [Acta Math. **72** (1940), 15–29; MR **1**, 297] and Bang [Copenhagen thesis, 1946; MR **8**, 199], and also to show that every infinitely differentiable function on an infinite interval is the sum of two functions each of which has a sequence of derivatives which tend to zero uniformly on compact sets, and in particular is the sum of two Denjoy-Carleman quasi-analytic functions. *R. P. Boas, Jr.*

Bruhat, François; et Cartan, Henri. Sur les composantes irréductibles d'un sous-ensemble analytique-réel. C. R. Acad. Sci. Paris **244** (1957), 1123–1126.

Continuing the study of real-analytic manifolds with singularities [same C. R. **244** (1957), 988–990; MR **19**, 125] the authors introduce for a submanifold the concepts of K-irreducibility, for any compact set K, which is such that a first result is as follows. If g is an irreducible germ at a point a, and $\hat{g}$ is the intersection of all submanifolds containing g, then $\hat{g}$ is K-irreducible for any K. The authors then introduce and make statements, about (irreducible) K-components of a submanifold, and state interesting examples. At the end they offer the following undecided question (and a second one too): If g and $\hat{g}$ are as above, and g is a component germ of $\hat{g}$, can $\dim \hat{g} > \dim g$? *S. Bochner* (Princeton, N.J.).

See also: Approximations, Orthogonal Functions: Morozov. **Trigonometric Series and Integrals:** Wintner. **Topological Vector Spaces:** Cotlar.

Measure, Integration

Poprużenkó, J. Sur une propriété d'une classe de mesures abstraites. Colloq. Math. **4** (1957), 189–194.

Let $I(n)$ denote the following hypothesis: There does not exist an inaccessible aleph $\leqq n$. The reviewer has proved [Fund. Math. **20** (1933), 221–223] that if the hypothesis $I(2^{\aleph_0})$ is true, every set of positive outer Lebesgue measure contains a non-denumerable number of subsets all disjoint and of positive outer measure. Sierpiński [ibid. **23** (1934), 125–134] deduced from this the following theorem. Under the hypothesis $I(2^{\aleph_0})$, every linear, non-denumerable set E contains a non-denumerable infinity of disjoint sets, each of which has an outer Lebesgue measure equal to the measure of E. The author generalizes this theorem in proving that every non-denumerable set has a similar property relative to a vast class of abstract measures. *S. Ulam* (Los Alamos, N.Mex.).

Fabian, Václav. L'extension d'une mesure au σ-corps contenant chaque sousensemble composé d'un seul point. Časopis Pěst. Mat. **82** (1957), 308–313. (Russian. Czech and French summaries)

Soit μ une mesure (σ-additive mais pas nécessairement finie) dans un σ-corps $\mathscr{S}$. Soit $A(x)=\bigcap\{A;\ x\in A\in\mathscr{S}\}$ pour chaque $x\in\bigcup\mathscr{S}$, soit $\mathscr{A}=\{A(x);\ x\in\bigcup\mathscr{S}\}$, $\mathscr{N}=\mathscr{A}-\mathscr{S}$. Soit f une fonction définie dans $\mathscr{N}$, $0\leqq f(A)\leqq\mu(B)$ pour chaque $A\in\mathscr{N}$, $A\subset B\in\mathscr{S}$; soit finalement $\sum_{A\in\mathscr{N}} f(A)<+\infty$.

Il existe des mesures ν et ν_0, extensions de μ, définies dans les σ-corps $\mathscr{S}_\nu$ et $\mathscr{S}_{\nu_0}$ respectivement, et satisfaisant à ces conditions-ci: 1. Pour chaque $x\in\bigcup\mathscr{S}$ nous avons $\bigcap\{A;\ x\in A\in\mathscr{S}_\nu\}=A(x)\in\mathscr{S}_\nu$. 2. Si $\mu(\emptyset)=0$, $\nu(A)=f(A)$ pour chaque $A\in\mathscr{N}$. 3. Pour chaque $x\in\bigcup\mathscr{S}$, l'ensemble $\{x\}$, contenant un seul point x, appartient à $\mathscr{S}_{\nu_0}$.

Résumé de l'auteur.

Mařík, Jan. On the measures of Baire and Borel. Časopis Pěst. Mat. **81** (1956), 431–450. (Czech)

This paper is largely expository, but also contains some new results. Let P be a topological space, $\mathfrak{G}$ the family of open subsets of P, $\mathfrak{G}^*$ the family of P of the form $\{x: x\in P, f(x)>0\}$ for a continuous real-valued function f on P. The σ-algebra of subsets of P generated by $\mathfrak{G}$ ($\mathfrak{G}^*$) is denoted by $\mathfrak{B}$ ($\mathfrak{B}^*$) and is called the family of Borel (Baire) sets in P. Let μ be a countably additive, non-negative measure on $\mathfrak{B}^*$ (a Baire measure), such that $\mu(\emptyset)=0$. Let $\mathfrak{P}=\mathfrak{P}(\mu)$ be the family of all $A\subset P$ such that there are sets $G_n\in\mathfrak{G}^*$ ($n=1, 2, 3, \cdots$) with the property that $A\subset\bigcup_{n=1}^{\infty} G_n$ and $\mu(G_n)<\infty$ for all n. A Baire measure μ is said to have the property V_P if $\mu(B)<\infty$ implies that $B\in\mathfrak{P}$. A Baire measure μ is said to have property W_P if there is a Borel measure λ on P such that: 1) $B\in\mathfrak{B}^*$ implies that $\lambda(B)=\mu(B)$; 2) $G\in\mathfrak{G}\cap\mathfrak{P}$ implies that $\lambda(G)=\sup\{\mu(H): H\in\mathfrak{B}^*, H\subset G\}$; 3) $B\in\mathfrak{B}$ and $B\notin\mathfrak{P}$ imply $\lambda(B)=\infty$; 4) $B\in\mathfrak{B}$ implies that $\lambda(B)=\inf\{\lambda(G): G\in\mathfrak{G}, G\supset B\}$. Theorem: Let P be a completely regular space and let μ be a Baire measure with property V_P. For every set F of the form G' with $G\in\mathfrak{G}^*$, suppose that there are compact sets $K_1, K_2, \cdots$, such that $\sup\{\mu(H): H\in\mathfrak{B}^*, H\subset F\cap(\bigcup_{n=1}^{\infty} K_n)'\}=0$. Then μ has property W_P. Theorem: Let P be normal and countably paracompact or Hausdorff and paracompact. Then every Baire measure with property V_P also has property W_P. *E. Hewitt.*

Mařík, Jan. The Baire and Borel measure. Czechoslovak Math. J. **7(82)** (1957), 248–253. (Russian summary)

This paper is largely a reprise of the paper reviewed above with a few additions. Among these is an extension of a result of the reviewer [Fund. Math. **37** (1950), 161–189; MR **13**, 147]. The substance of Remark 1, p. 170, is extended from all perfectly normal spaces to all normal countably paracompact spaces. *E. Hewitt.*

Mařík, Jan. Les fonctionnelles sur l'ensemble des fonctions continues bornées, définies dans un espace topologique. Studia Math. **16** (1957), 86–94.

Let P be a topological space, $C^*(P)$ the linear space of all bounded real-valued continuous functions on P. A subset A of P is called relatively pseudo-compact if every continuous real-valued function on P is bounded on A. Let $\mathfrak{A}$ be a non-void family of subsets of P, and $\mathfrak{B}_\infty(\mathfrak{A})$ the family of all subsets of $C^*(P)$ of the form $\{f: f\in C^*(P), |f(x)|<a_n$ for all $x\in A_n$ $(n=1, 2, \cdots)\}$; here $\{A_n\}_{n=1}^{\infty}\subset\mathfrak{A}$ and $\lim_{n\to\infty} a_n=\infty$. Let $C^*(P)$ be topologized by taking all of the sets in $\mathfrak{B}_\infty(\mathfrak{A})$ as neighborhoods of the function 0. Theorem 1: Let J be a linear functional on $C^*(P)$, continuous in the topology on $C^*(P)$ induced by $\mathfrak{B}_\infty(\mathfrak{A})$, where all of the sets in $\mathfrak{A}$ are relatively pseudo-compact. Then J can be represented by an integral with respect to a countably additive Baire measure on P. Theorem 3: Suppose that P is a locally compact, σ-compact, Hausdorff space. Let J be a linear functional on $C^*(P)$ such

that $J(f_n)\to 0$ for every sequence $\{f_n\}$ of functions in $C^*(P)$ that converges uniformly to zero on every relatively pseudo-compact set. Then there is a compact Baire subset K of P and a countably additive Baire measure ϕ with support K such that J is the integral with respect to ϕ. Theorem 2 as stated in the paper depends in part upon Lemma 1, the proof of which is incorrect. (The last inequality on p. 87 may fail, as simple examples show.) Hence Theorem 2 is in doubt. The novelty of the present paper lies in the conclusions that the measures obtained are countably additive, since a theorem due essentially to A. A. Markov [see Hewitt, Ark. Mat. **2** (1952), 269–282; MR **14**, 545] shows that J, under certain weak hypotheses, can be written as the integral with respect to a finitely additive Baire measure. [For earlier work by the author on the same theme, see Czechoslovak Math. J. **5**(80) (1955), 467–487; MR **19**, 256.] *E. Hewitt.*

Mařík, Jan. La transformation des intégrales simples. Časopis Pěst. Mat. **82** (1957), 93–98. (Russian. Czech and French summaries)

Soit f une fonction continue à variation bornée dans l'intervalle $\langle a, b\rangle$. Soient p, n, v les variations positive, négative et totale de la fonction f. Définissons les fonctions ρ, $\varkappa$, η dans l'intervalle $\langle a, b\rangle$ de la manière suivante: Si $x\in(a, b)$ et si la fonction f est croissante (décroissante) au point x, posons $\rho(x)=\eta(x)=1$, $\varkappa(x)=0$ ($\rho(x)=0$, $\varkappa(x)=1$, $\eta(x)=-1$); dans le reste de l'intervalle $\langle a, b\rangle$ soit $\rho(x)=\varkappa(x)=\eta(x)=0$. Soit F (resp. Φ) une fonction définie sur l'ensemble $\langle a, b\rangle$ (resp. $f(\langle a, b\rangle)$). Dans ce cas, l'ensemble $f^{-1}(y)$ est fini pour presque chaque $y\in E_1$ et les relations

$$\int_{E_1}\Big(\sum_{f(x)=y}F(x)\rho(x)\Big)dy=\int_a^b F(x)dp(x),$$

$$\int_{E_1}\Big(\sum_{f(x)=y}F(x)\varkappa(x)\Big)dy=\int_a^b F(x)dn(x),$$

$$\int_{E_1}\Big(\sum_{f(x)=y}F(x)\eta(x)\Big)dy=\int_a^b F(x)df(x),$$

$$\int_{E_1}\Big(\sum_{f(x)=y}F(x)|\eta(x)|\Big)dy=\int_a^b F(x)dv(x),$$

$$\int_{E_1}\Big(\sum_{f(x)=y}F(x)\Big)dy=\int_a^b F(x)dv(x),$$

$$\int_{f(a)}^{f(b)}\Phi(y)dy=\int_a^b\Phi(f(x))df(x)$$

sont valables au sens suivant: Si l'intégrale (finie ou infinie) au second membre existe au sens de Lebesgue-Stieltjes, alors la somme sous le signe de l'intégration au premier membre possède un sens pour presque tout $y\in E_1$, l'intégrale au premier membre existe au sens de Lebesgue et est égale à l'intégrale au second membre.

Résumé de l'auteur.

Revuz, André. Fonctions croissantes et mesures sur les espaces topologiques ordonnés. Ann. Inst. Fourier, Grenoble **6** (1955–1956), 187–269.

Let F be a positive increasing function continuous on the right defined on an ordered topological space X. In Chapter I the author finds conditions on X and F in order that the set $C_-(x_0)=\{x\in X\mid x\leqq x_0\}$ is always measurable with measure $F(x_0)$ for a positive Radon measure on X and for other natural interpretations of a measure for subsets of X. In Chapter II the values of F are taken to be in a topological ordered group Y and a definition of bounded variation is given for F. Under appropriate conditions, the most severe being that the sets $C_-(x)$ are compact, $F(x)$ is equal to the value on $C_-(x)$ of a Radon measure on X if and only if F has bounded variation and is continuous on the right. In some cases the condition on $C_-(x)$ can be relaxed. In Chapter III a theory of integration with respect to these F is given. The cases where X is the set of all non-empty compact subsets of a locally compact space ordered under set inclusion and where X is the space of all continuous real functions on a compact space are treated in detail. If X has sufficient structure there is a theorem of the F. Riesz type. Any linear functional M with values in a space Y, relatively bounded on the space of continuous real functions vanishing outside compact subsets of X, is given by $M(\phi)=\int\phi dF$, where F has values in Y, is of bounded variation, and is continuous on the right. *B. Yood* (Eugene, Ore.).

Pauc, Chr.; et Revuz, A. Sur une formule générale d'intégration par parties. Ann. Mat. Pura Appl. (4) **42** (1956), 307–312.

Sia X uno spazio topologico ordinato nel senso di Revuz [vedi l'articulo recensito precedente]. Sia $C_-(x)$ l'insieme di tutti i punti che precedono x. Si ponga $S(x, u_1, \cdots, u_n)=C_-(x)-\mathbf{U}_{i=1}^n C_-(u_i)$. Se $U(x)$ è una funzione reale definita in X, si pone

$$\Delta(S)=U(x)-\textstyle\sum_1 U(\inf xu_i)+\cdots$$
$$+(-1)^p\textstyle\sum_p U(\inf xu_{i_1}\cdots u_{i_p})+\cdots$$
$$+(-1)^n\textstyle\sum_n U(\inf xu_{i_1}\cdots u_{i_n}).$$

La $U(x)$ dicesi a variazione limitata se per ogni S di X si ha: $\sup\sum_{i=1}^n|\Delta(S_i)|<+\infty$, essendo $S=\mathbf{U}_{i=1}^n S_i$, $S_i\cap S_j=\emptyset$. Associando ad una funzione a variazione limitata e continua a destra una opportuna misura di Radon, gli Autori dimostrano una formola di integrazione per parti in uno spazio topologico avente due ordinamenti, l'uno duale dell'altro. Gli Autori approfittano di questo lavoro per apportare una correzione ad un lavoro precedente di Pauc [Ann. Mat. Pura Appl. (4) **40** (1955), 183–192; MR **17**, 1190]. *G. Fichera* (Roma).

Eberlein, W. F. Notes on integration. I. The underlying convergence theorem. Comm. Pure Appl. Math. **10** (1957), 357–360.

Es sei I ein Integral in einem kompakten Hausdorffschen Raum S, d.h., ein lineares positives Funktional über dem Raum C der reellen stetigen Funktionen auf S. Der Satz, daß bei einer gleichmäßig beschränkten, punktweise gegen eine Funktion f aus C konvergierenden Folge (f_n) in C gilt $\lim_n I(f_n)=I(f)$, wird sehr einfach direkt, d.h., als Satz über stetige Funktionen allein und ohne Ausdehnung von I, bewiesen, wobei die L^1-, L^2- und L^∞-Normen auf C Verwendung finden, und es wird erläutert, warum ein solcher Beweis wünschenswert sei.

K. Krickeberg (Würzburg).

Bellman, Richard. On a generalization of the Stieltjes integral to matrix functions. Rend. Circ. Mat. Palermo (2) **5** (1956), 181–186.

Let $x(t)$ be an $n\times n$ matrix function on $I=[0, 1]$ such that $x_i\equiv x(t_{i+1})-x(t_i)$ is non-negative definite for any $0\leqq t_i<t_{i+1}\leqq 1$; let $\lambda_1(t_i)\geqq\cdots\geqq\lambda_n(t_i)\geqq 0$ be the eigenvalues of x_i; and choose successive subdivisions of I such as those defined by the points $t_0=0$, $t_i=t_{i-1}+2^{-N}$ ($1\leqq i\leqq 2^N$; $N=1, 2, \cdots$). The author proves the following theorems. (I) If $f(t)$ is a continuous (scalar) function on I, then, for any k ($1\leqq k\leqq n$), $\sum_0^{N-1} f(t_i)\lambda_k(t_i)$ tends as $N\to\infty$ to a linear functional, denoted by $\int_0^1 f(t)d\lambda_k$. (II) If $x(t)$ is a

2×2 continuous, as well as monotone increasing, matrix function on I, y_t the (unique) non-negative definite square root of x_t, and $F(t)$ a continuous (2×2) matrix function on I, then $\sum_0^{N-1} y_t F(t_t) y_t$ tends as $N\to\infty$ to a linear matrix functional, denoted by $\int_0^1 \sqrt{(dx)}F(t)\sqrt{dx}$. The proof of (I) uses a result of Ky Fan [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 652–655; MR **11**, 600]. The proof of (II) rests upon establishing the convergence of $\sum_0^{N-1}\sqrt{[\lambda_1(t_t)\lambda_2(t_t)]}$ as $N\to\infty$ (hence the restriction to $n=2$).

H. A. Antosiewicz (Providence, R.I.).

See also: Calculus of Variations: Young; Sigalov. **General Topology:** Sternberg. **Probability:** Kampé de Fériet; Ryll-Nardzewski.

Functions of Complex Variables

Trohimčuk, Yu. Yu. On the theory of differentiation of functions of a complex variable. Ukrain. Mat. Ž. **8** (1956), 177–190. (Russian)

The set $\mathfrak{M}_z$ of all derivative numbers at the point z is called a set of monogenicity. This article examines the connection between sets of monogenicity and the differentiability properties of the functions $u(x, y)$ and $v(x, y)$. If the functions $u(x, y)$ and $v(x, y)$ are differentiable at a certain point $z=x+iy$, then the set $\mathfrak{M}_z$ of the function $f(z)=u+iv$ is a circumference or a point. If $f(z)$ is continuous in the region D and at every point of the set $M\subseteq D$ at least one of the functions u, v is differentiable, then at almost every point $z\in M$ the set $\mathfrak{M}_z$ is a circumference (in particular, a point) or else the whole plane. For an arbitrary continuous function $f(z)$, at almost every point z of the region either $\mathfrak{M}_z$ is the whole plane or else $f(z)$ is asymptotically differentiable (and consequently $\mathfrak{M}_z$ is "asymptotically" a circumference). If the set $\mathfrak{M}_z$ of the function $f(z)$ does not depend on z, then either $\mathfrak{M}_z$ is the whole plane or else $\mathfrak{M}_z$ is a circumference (in particular, a point); in the latter case the function $f(z)$ has the form

$$f(z)=Az+B\bar{z}+C.$$

In Section 6 the differentiability properties are studied of a function $f(z)$ which is analytic in a region D and can be extended continuously on the everywhere discontinuous perfect set $P\subset D$ of its singular points.

A. D. Taĭmanov (RŽMat **1957**, no. 4767).

Bilimović, Anton. Sur la mesure d'une fonction non-analytique par rapport à une fonction analytique. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N.S.) **9** (1956), 1–11. (Serbo-Croatian. French summary)

For a complex function $f(z)=P(x, y)+iQ(x, y)$ the expression $\bar{B}=\operatorname{grad} Q-\bar{k}\times\operatorname{grad} P$, where $\bar{k}$ is the unit vector normal to the z-plane, vanishes whenever $f(z)$ is analytic, and, in the general case, serves as a measure of how far $f(z)$ deviates from being analytic. The author considers the classes of functions obtained by imposing the conditions $B_1=0$ or $B_2=0$, where B_1 and B_2 are the components of $\bar{B}$ having the same direction as grad P or normal to grad P, respectively. As the author pointed out earlier [Acad. Serbe Sci. Publ. Inst. Math. **6** (1954), 17–26; MR **16**, 236], the second class contains the para-analytic functions of Fréchet. *A. J. Lohwater.*

Bilimović, Anton. Sur la transformation affine d'une fonction non-analytique à la fonction analytique. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N.S.) **9** (1956), 13–17. (Serbo-Croatian. French summary)

Bilimović, Anton. Sur le diagramme d'une fonction non-analytique pour un point donné. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N. S.) **9** (1956), 39–43. (Serbo-Croatian. French summary)

Morev, I. A. On a generalization of the concept of monogenic functions. Mat. Sb. N.S. **42(84)** (1957), 197–206. (Russian)

The generalization ("monogenic functions of several hypercomplex variables") is that introduced by the author [Ukrain. Mat. Ž. **8** (1956), 423–434; MR **19**, 400]. The author obtains the general form of a monogenic function for the algebra of dual numbers in the one- and two-variable cases. Here the underlying algebra has basis elements $e_1=1$, $e_2=\omega$, with $\omega^2=0$; $\zeta=a^1+\omega a^2$, where a^j are complex-valued functions over a region D in Euclidean n-space. Then $f=f\{\zeta; D\}$ is monogenic with respect to ζ if and only if f has the form

$$\phi\{a^1\}+\omega\left[\frac{d\phi}{da^1}\cdot a^2+H\{a^1\}\right],$$

where $\phi\{a^1\}$, $H\{a^1\}$ denote suitable functions of a^1 in a certain neighborhood of each point of D; and then

$$f\zeta=\frac{d\phi}{da^1}+\omega\left[\frac{d^2\phi}{(da^1)^2}\cdot a^2+\frac{dH}{da^1}\right].$$

Next let $f=\phi+\omega\Phi$; a necessary and sufficient condition for f to be monogenic with respect to ζ and η, over a region D in (x, y, z, t)-space, where $\zeta=\alpha+\omega\beta$, $\eta=p+\omega q$, $\alpha=x+iy$, $p=z+it$, is that

$$f=\phi\{\alpha, p\}+\omega\left[\frac{\partial\phi}{\partial\alpha}\cdot\beta+\frac{\partial\phi}{\partial p}\cdot q+H\{\alpha, p\}\right],$$

with similar notation and conditions. If D is a bicylinder then in a suitable sub-bicylinder the following Taylor expansion holds:

$$f(N)=\Sigma\frac{1}{k!}\left(\frac{\partial}{\partial\zeta}\Delta\zeta+\frac{\partial}{\partial\eta}\Delta\eta\right)^{(k)}[f(M)].$$

E. S. Pondiczery (Princeton, N.J.).

★ **Amato, Vincenzo. Funzioni di matrice.** Scritti matematici in onore di Filippo Sibirani, pp. 5–11. Cesare Zuffi, Bologna, 1957.

Es sei y eine konstante Matrix von der Ordnung n, x ihre Weierstrass'sche Normalform, $x=u^{-1}yu$. Ferner sei $f(z)=\sum_{n=0}^{\infty}(f^{(k)}(\alpha)/k!)(z-\alpha)^k$ eine holomorphe Funktion von z und $F(y)=\sum_{k=0}^{\infty}(f^{(k)}(\alpha)/k!)(y-\alpha j_n)^k$ die entsprechende Matrix (j_n stellt die Einheitsmatrix von der Ordnung n dar). Man nimmt an, dass die in den n^2 Elementen von $F(y)$ auftretenden Reihen konvergent sind. Sodann gilt die Formel $F(y)=uf(x)u^{-1}$, wobei $f(x)$ eine vom Verfasser genau beschriebene Matrix von der Ordnung n bedeutet. Dieselbe hängt von den Werten der Funktion $f(z)$ und ihrer Ableitungen sowie von den Elementarteilern der Matrix y ab. *O. Borůvka* (Brno).

Redheffer, Raymond. The fundamental theorem of algebra. Amer. Math. Monthly **64** (1957), 582–585.

The author uses the barest elements of real analysis, in combination with the Cauchy-Riemann equations, to establish a convenient version of the minimum-modulus principle. This principle, together with the elementary fact that the modulus of a non-constant polynomial actually assumes its minimum value, demonstrates the existence of a zero. *Bernard Epstein* (Stanford, Calif.).

Belinskiĭ, P. P. On variations of quasiconformal mapping. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 93–95. (Russian)

Our aim is to find approximate values for quasiconformal mappings with characteristics $p(z)$, $\theta(z)$ close to the characteristics of a conformal mapping, that is, for which $p(z)-1$ is small. *From the introduction.*

Grunsky, Helmut. Über konforme Abbildungen, die gewisse Gebietsfunktionen in elementare Funktionen transformieren. I. Math. Z. **67** (1957), 129–132.

The author considers a domain $\mathfrak{z}$ of connectivity n with non-degenerate boundary components $Z_1, \cdots, Z_n$ which may be taken as analytic Jordan curves. He takes points $z_i \in \mathfrak{z}$ $(i=1, \cdots, k)$; real constants α_i $(i=1, \cdots, k)$, c_ν $(\nu=1, \cdots, n)$; denotes the Green's function of $\mathfrak{z}$ with pole at z_i by $g(z, z_i)$, the harmonic measure of Z_ν by $h_\nu(z)$; and forms $g(z)=\sum_{i=1}^{k} \alpha_i g(z, z_i)+\sum_{\nu=1}^{n} c_\nu h_\nu(z)$. Then under the assumption that $\partial g/\partial n$ has no change of sign on a boundary component Z_ν he proves the following result. Assuming as a normalization that $\mathfrak{z}$ contains the point at infinity, which is to be distinct from all z_i, there exists a unique conformal mapping $w=F(z)$ with expansion at infinity $F(z)=z+O(1/z)$ of $\mathfrak{z}$ into the w-sphere such that for $\phi(w)$ inverse to $F(z)$ we have $g(\phi(w))=\log|S(w)|$, where $S(w)=A\prod_{i=1}^{k}(w-w_i)^{-\alpha_i}\prod_{\nu=1}^{n}(w-b_\nu)^{-\beta_\nu}$, with $A=e^{g(\infty)}$, $w_i=F(z_i)$, $\beta_\nu=(2\pi)^{-1}\int_{Z_\nu}\partial g/\partial n\, ds$. Further, exactly one point b_ν lies in each domain complementary to $F(\mathfrak{z})$. The method of proof is to complete $\mathfrak{z}$ suitably to an elliptic Riemann surface and map the latter conformally onto the w-sphere, from which the asserted properties readily follow. From this result one readily derives the existence and essential uniqueness of certain canonical conformal mappings recently given by Walsh [Trans. Amer. Math. Soc. **82** (1956), 128–146; MR **18**, 290].

{This result is a special case of one given some years ago by the reviewer and D. C. Spencer in Ann. of Math. (2) **53** (1951), 4–35, § 8 [MR **12**, 400]. There it is shown that every positive quadratic differential of the domain $\mathfrak{z}$ can be imbedded in a hyperelliptic quadratic differential so that the complementary domains lie in circle domains associated with the trajectory structure of the latter. The corresponding uniqueness result, although easily derived, was not given there explicitly. In the present special case one can operate even with linear differentials. The reviewer has independently given a detailed exposition of this aspect in the case of Walsh's mappings [see review second below]. The two proofs are on precisely the same lines although in different notations. While the reviewer's paper explicitly referred to domains of genus zero, similar results are readily given for domains of finite genus.} *J. A. Jenkins.*

Grunsky, Helmut. Über konforme Abbildungen, die gewisse Gebietsfunktionen in elementare Funktionen transformieren. II. Math. Z. **67** (1957), 223–228.

The author extends the result of the paper reviewed above in that he drops the condition there imposed that $\partial g/\partial n$ has no change of sign on a boundary component Z_ν. The method is as before to complete the domain $\mathfrak{z}$ to an elliptic Riemann surface but now with the imposition of suitable boundary identifications. Uniqueness of the mapping no longer holds in all cases. He derives from this result a well-known theorem of Julia and indicates another application. The author's extension does not coincide with that given by the reviewer and Spencer [loc. cit., preceding review]; however, one can recognize further extensions which include both and go considerably beyond them. *J. A. Jenkins.*

Jenkins, James A. On a canonical conformal mapping of J. L. Walsh. Trans. Amer. Math. Soc. **88** (1958), 207–213.

The author remarks that Walsh's canonical conformal mappings and the corresponding uniqueness theorems can be deduced quickly from results by Jenkins and Spencer on quadratic differentials [for references see the second preceding review]. He gives a detailed exposition "from the beginning" in which he actually has to use only linear differentials because of the relatively special nature of the situation under investigation. The same proofs were given simultaneously by Grunsky [second preceding review]. *R. P. Boas, Jr.* (Evanston, Ill.).

Fil'čakov, P. F. The method of successive conformal mappings and its applications to filtration problems. I. Ukrain. Mat. Ž. **7** (1955), 453–470 (1 insert). (Russian)

Let D_n be a domain on the complex half-plane $\operatorname{Im} z_n<0$ whose boundary consists of a part of the real axis and an arc of a half-ellipse. Let E_n be a conformal mapping of D_n onto a half-plane z_{n+1}, $\operatorname{Im} z_{n+1}<0$, $(n=1, 2, \cdots)$. The formula for E_n is given. If the boundary of the given domain D contains several arcs of half-ellipses (the D-type domain), then the successive application of conformal mapping of E_n-type, gives a mapping of D onto a half plane $\operatorname{Im} z<0$. The author approximates a given domain D situated on the lower half-plane, whose boundary contains parts of the real axis, by domains of D-type mentioned above. This method can be applied to some practical problems connected with the problem of filtration.

J. Górski (Zbl **67** (1957), 58).

Fil'čakov, P. F. The method of successive conformal mappings and its application to filtration problems. II. Case of an arbitrary curve for a dam. Ukrain. Mat. Ž. **8** (1956), 76–91 (3 inserts). (Russian)

The calculation of a dam, implanted on a permeable soil, leads the author to solve the problem of the conformal representation of a half-plane on an asymmetric polygonal domain, of which two of the sides are constituted by rectilinear cuts. The formula of Christoffel-Schwarz gives immediately the indeterminate solution of the problem. The author calculates approximately the arbitrary parameters of the formula as functions of the dimensions of the polygon.

The conformal representation (normalised by $dz/d\zeta=1$ at infinity) of $\Im(\zeta)>0$ on the region consisting of $\Im(z)>0$ together with (or less) an isosceles triangle having its base on $\Im(z)=0$ is first considered. The Christoffel-Schwarz integral in this case becomes an incomplete beta function and the lengths of the segments of the real axis (in the ζ-plane) corresponding to the sides of the triangle can be expressed in terms of gamma functions. This relationship is given in graphical form. (Fig. 2. The fifth vertical division, not the fourth, should be marked 1.0). Various approximations to this conformal representation are obtained in power series form.

This representation is applied repeatedly to reduce the region above an arbitrary curve (which is supposed to lie near the real axis and coincide with the real axis at its ends) gradually towards the upper half-plane.

The necessary calculations are given in tabular and in graphical form for a special case. *A. J. Macintyre.*

Fil'čakov, P. F. Method of successive conformal mappings. III. Ukrain. Mat. Ž. **8** (1956), 299–318. (Russian)

For certain geometrical configurations the procedure previously investigated [see the two papers reviewed above] is inadequate in convergence and available information for purposes of computation. The method is extended here by the use of the conformal representation of the half plane $\Im(z)<0$, cut by a slit from an arbitrary interior point to the origin, on a second half plane. This conformal representation is investigated by means of the Christoffel-Schwarz integral and various power series expansions. The analysis is developed so far as to provide numerical estimates in a case involving two vertical slit-like obstructions in a horizontal bed.

A. J. Macintyre (Aberdeen).

Volkovyskiĭ, L. I. Contemporary investigations in the theory of Riemann surfaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 77–84. (Russian)

An account is given of a survey report on certain questions in the contemporary theory of Riemann surfaces, presented by the author at a conference on the theory of functions of a complex variable at Moscow State University. The author deals with the following questions: 1) the concept of a Riemann surface; 2) functions and differentials on Riemann surfaces; 3) classification of Riemann surfaces; 4) topological and variational methods on Riemann surfaces. Some still incompletely solved problems are formulated. There is a bibliography of 39 entries. *G. C. Tumarkin* (RŽMat **1957**, no. 4757).

af Hällström, Gunnar. Automorphiefunktionen, Streckenkomplex, Typus. S.-B. Berlin. Math. Ges. **1954/55–1955/56**, 37–38.

Summary of Acta Acad. Abo. **20** (1956), no. 10 [MR 18, 883].

Erdös, P.; and Rényi, A. On the number of zeros of successive derivatives of entire functions of finite order. Acta Math. Acad. Sci. Hungar. **8** (1957), 223–225.

The authors sharpen one of their results [same Acta **7** (1956), 125–144; MR **18**, 201] for entire functions of finite order at least 1. The theorem reads: if f is an entire function of finite order $\rho\geqq 1$, if $M(r)$ is its maximum modulus, and $H(y)$ is the inverse of $y=\log M(x)$; and if N_k is the number of zeros of $f^{(k)}(z)$ in $|z|<1$; then

$$\liminf k^{-1}N_k H(k) < e^{2-1/\rho}.$$

R. P. Boas, Jr. (Evanston, Ill.).

Rényi, Catherine. On periodic entire functions. Acta Math. Acad. Sci. Hungar. **8** (1957), 227–233.

Let $Z_a(n)$ denote the number of vanishing coefficients (among the first $n+1$ coefficients) in the Taylor series of the transcendental entire function f about $z=a$. If $a\neq b$, put $\Delta(n)=Z_a(n)+Z_b(n)-n$. In her earlier paper [same Acta **7** (1956), 145–150; MR **18**, 201] the author showed that $\lim n^{-1}\Delta(n)\leqq 0$. Here she shows first that this can be improved if f is of finite order $\rho\geqq 1$, as follows: $\liminf n^{-1+1/(\rho+\varepsilon)}\Delta(n)\leqq 0$ for every $\varepsilon>0$; if f is of finite type τ then $\liminf n^{-1+1/\rho}\Delta(n)\leqq \frac{1}{2}|b-a|e^2(\tau/e)^{1/\rho}$. Next the author shows that the lower density of the vanishing coefficients of the power series of a periodic entire function cannot exceed $\frac{1}{2}$, with corresponding improvements for functions of finite order. From this she deduces that if f is any integrable function of period 2π the lower density of the vanishing coefficients in its Hermite expansion does not exceed $\frac{1}{2}$. Finally she constructs a periodic entire function for which $\Delta(n)\to\infty$. *R. P. Boas, Jr.*

Džrbašyan, M. M. On integral representation and expansion in generalized Taylor series of entire functions of several complex variables. Mat. Sb. N.S. **41(83)** (1957), 257–276. (Russian)

Here the order and type of an entire function of two complex variables are defined in the same fashion as for one variable by using $M(r)=\max|f(z_1,z_2)|$ for $|z_1|+|z_2|=r$. For functions of finite order the author obtains generalized Pólya integral representations analogous to the one he found for the one-variable case [Izv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 225–252; MR **14**, 33]. Using these he obtains theorems on the representation of a function by the two-dimensional analogue of the Abel-Goncharov series, i.e., a series of the form

$$\sum\sum f^{(p,q)}(\alpha_{pq},\beta_{pq})A_{pq}(z_1,z_2),$$

where A_{pq} are polynomials. *R. P. Boas, Jr.*

Dinghas, Alexander. Zum Verhalten eindeutiger analytischer Funktionen in der Umgebung einer wesentlichen isolierten Singularität. Math. Z. **66** (1957), 389–408.

The author proves several theorems concerning meromorphic functions defined on a punctured disc and having an essential singularity at the point. These theorems are sharp forms of the so-called Weierstrass' theorem on the behavior of such functions near their essential singularity.

First, he proves: If $w=w(z)$ is a function meromorphic in $|z|<\infty$ and if $0<\alpha<1$, then, for an arbitrary a, the set $\{r|\ \mu(r,a)<\exp(-\alpha T(r))\}$ has no upper bound. $\mu(r,a)$ is the chordal distance between a and the image of $|z|=r$, and $T(r)$ is Nevanlinna's characteristic function of $w(z)$. Here, a new proof for the equivalence of $T(r)=O(\log r)$ and rationality is given without making use of the canonical product representation of $w(z)$.

Next, he localizes the above theorem by constructing the "local" theory corresponding to Nevanlinna's "global" theory [cf. G. af Hällström, Acta Acad. Abo. **12** (1940), no. 8; MR **2**, 275]: If $w=w(z)$ is a function meromorphic in $r_0<|z|<\infty$ and if $0<\alpha<1$, then, for an arbitrary a, the set $\{r|\ [w(z),a]<\exp(-\alpha T(r))$ has no upper bound. $T(r)$ is the "local" characteristic function $w(z)$ in his own sense, and $[w_1,w_2]$ expresses the chordal distance.

Further sharp forms are obtained and, moreover, a theorem corresponding to Ahlfors' theorem on covering surface and Nevanlinna's defect relation are proved for the above-mentioned functions. *M. R. Hestenes.*

Daĭovitch, Voĭn. Existence des valeurs limites des résultantes de certaines classes de fonctions analytiques. Bull. Soc. Math. Phys. Serbie **8** (1956), 147–156. (Serbo-Croatian. French summary)

En exprimant la résultante $h(z)=\sum a_\nu b_\nu z^\nu$ de deux fonctions analytiques $f(z)=\sum a_\nu z^\nu$ et $g(z)=\sum b_\nu z^\nu$ $(\nu\geqq 1)$ régulières pour $r=|z|<1$ par la formule bien connue de Parseval $h(re^{i\theta})=\pi^{-1}\int_0^{2\pi}\mathrm{Re}\,g(\rho e^{i\varphi})f(\rho e^{i(\theta-\varphi)})d\varphi$ $(r=\rho^2<1)$, l'auteur déduit de là la conclusion évidente que $|h(re^{i\theta})|$ sera borné si $|f|$ et $\int_0^{2\pi}|\mathrm{Re}\,g(\rho e^{i\varphi})|d\varphi$ sont uniformément bornés, ou inversement. D'après le théorème classique de Fatou on conclut alors, que sous ladites conditions, les valeurs limites radiales de $h(re^{i\theta})$ existent presque partout. *M. Tomić* (Beograd).

Bernardi, S. D. A determinant inequality for univalent functions. Amer. Math. Monthly **64** (1957), 495–497.

Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ be regular and univalent in

$|z|<1$, and define $b_n(\alpha)$ by $(z/f(z))^{\alpha/2}=1+\sum_{n=1}^{\infty} b_n(\alpha)z^n$. The author proves several assertions about the coefficients $b_n(\alpha)$, and in particular finds the sharp bound for $|b_{n-1}(2n)|$. *A. W. Goodman* (Lexington, Ky.).

Biernacki, Mieczysław. Sur les coefficients de Taylor des fonctions univalentes. II. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 127–133 (1957). (Polish and Russian summaries)

[For Part I see Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 5–8; MR **17**, 957.] Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in $|z|<1$. The author proves that

$$n^{-1}\sum_{k=1}^{n} \big||a_{k+1}|-|a_k|\big|<C\log n \quad (n=2, 3, \cdots).$$

If in addition $z^p f'(z)$ is area p-valent in $|z|<1$, then

$$\big||a_n|-|a_{n-1}|\big|<C(p)\log n \quad (n=2, 3, \cdots).$$

The author shows further that the hypothesis that $f(z)$ be regular and univalent in the unit circle, does not imply that $z^p f'(z)$ is area p-valent there. He raises the question of determining additional hypotheses on $f(z)$ which will secure the conclusion. *A. W. Goodman.*

Čebotarev, G. N. Partial indices for the Riemann boundary-problem with a triangular matrix of second order. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 199–202. (Russian)

In an article by the reviewer [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 4(50), 3–54; MR **14**, 629] the conjecture was expressed that the partial indices of the boundary problem $\phi^+(t)=A(t)\phi^-(t)$ are closely connected with the indices of the characteristic functions of the matrix $A(t)$. In the present article it is shown that for a triangular matrix

$$A(t)=\begin{pmatrix}\xi_1(t) & 0\\ a(t) & \xi_2(t)\end{pmatrix}$$

the partial indices for the Riemann problem lie between the indices of the characteristic functions $\kappa_1=\operatorname{Ind}\xi_1(t)$ and $\kappa_2=\operatorname{Ind}\xi_2(t)$. The particular case is pointed out in which the partial indices coincide with the indices of the characteristic functions.

The work is based on the solution in closed form of the Riemann problem with triangular matrix and on expansion of certain functions into continued fractions.

F. D. Gahov (RŽMat **1957**, no. 3041).

Hvedelidze, B. V. On the discontinuous boundary problem of Riemann-Privalov with coefficients having critical points. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 40–43. (Russian)

The article deals with the boundary problem

$$(*)\qquad \Phi^+(t)=a(t)\Phi^-(t)+b(t)$$

under the conditions that the boundary Γ is a smooth curve, the inclination of whose tangent satisfies a Hölder condition, and that the coefficient $a(t)$ has the form

$$a(t)=\prod_{k=n_1+1}^{n}(t-c_k)^{\lambda_k}\prod_{k=n+1}^{m}\ln^{\mu_k}(t-c_k)\,a_*(t),$$

where c_k are given points of the boundary, λ_k, μ_k are given numbers, $a_*(t)$ is a non-vanishing function which satisfies the Dini condition everywhere except at a finite number of points $c_1, c_2, \cdots, c_n$, where it has a discontinuity of the first kind [cf. Hvedelidze, Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 81–88; MR **17**, 376].

Let $\sigma_k=[\lambda_k]$, with $\sigma_{n_1+1}, \cdots, \sigma_{n_2}<0$, $\sigma_{n_2+1}, \cdots, \sigma_n>0$. Also let $Q(z)$ be defined by

$$Q(z)=\begin{cases}\prod_{k=n_1+1}^{n_2}(z-c_k)^{\sigma_k}, & z\in E^+\\ \prod_{k=n_2+1}^{n}(z-c_k)^{\sigma_k}, & z\in E^-,\end{cases}$$

where E^+ and E^- are the finite and infinite regions bounded by Γ.

It is proved that there exists a solution of (*) of finite order at infinity in the class of functions

$$\Phi(z)=Q^{-1}(z)\Phi_1(z),\ \Phi_1(z)\in L_p(\Gamma, \omega(t)),$$

and an explicit expression is given for it in terms of integrals of Cauchy type.

The article is a generalization of work by a series of authors (Carleman, F. D. Gahov, N. I. Mushelišvili and others), who have considered problem (*) under more restrictive hypotheses.

F. D. Gahov (RŽMat **1957**, no. 6295).

Čikin, L. A. On stability of the Riemann boundary problem. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 44–46. (Russian)

The author considers the question of the stability of the solution of the Riemann boundary problem for a pair of analytic functions under deformation of the boundary. Here only such deformations are allowed as in their totality satisfy the conditions of Warschawsky [Math. Z. **35** (1932), 321–456].

The fundamental result obtained by the author is as follows. If L_0 is a plane smooth closed curve and $G(t)$ (coefficient in the boundary condition) satisfies a Hölder condition and does not vanish anywhere on L_0, then the solution of the homogeneous Riemann problem is stable. The proof depends on the theorem of Carathéodory about conformal mapping of regions with variable boundaries and on the theorem of Warschawsky. Analogous conditions are sufficient for the stability of a non-homogeneous Riemann problem.

A. V. Batyrev (RŽMat **1957**, no. 6298).

Gahov, F. D.; and Krikunov, Yu. M. Topological methods of the theory of functions of a complex variable and their application to inverse boundary problems. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 207–240. (Russian)

For given values of an analytic or pseudoanalytic function on the boundary of a region, relations are deduced for the number of critical points (zeros, poles, logarithmic points, pre-images of branch points) of the corresponding functions in the region. Applications of the relations obtained are given to the solution of inverse boundary problems for analytic functions.

Authors' summary.

Vekua, N. P. On a generalized boundary problem of Carleman for several unknown functions. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 377–384. (Russian)

Let S^+ be a finite region bounded by a simple closed Lyapunov curve L. Further, let there be given on L a function $\alpha(t)$ with non-vanishing derivative satisfying a Hölder condition. Also let $\alpha(t)$ map L in one-to-one fashion onto itself, whereby t and $\alpha(t)$ describe L in opposite directions and $\alpha[\alpha(t)]\equiv t$.

Using the theory of singular integral equations the

author gives a solution of the following boundary problem for analytic functions. To find a vector $\Phi(z)=(\Phi_1, \Phi_2, \cdots, \Phi_n)$ meromorphic in the region S^+ with the boundary condition

$$\Phi^+[\alpha(t)]=A(t)\Phi^+(t)+B(t)\Phi^+(t)+g(t),$$

where $\Phi^+(t)$ denotes the boundary values on L of the vector $\Phi(z)$ in S^+ and $A(t)=\|A_{kl}(t)\|$, $B(t)=\|B_{kl}(t)\|$ $(k, l=1, 2, \cdots, n)$ are given matrices satisfying a Hölder condition, while $g(t)=(g_1, g_2, \cdots, g_n)$ is a given vector also satisfying a Hölder condition. It is assumed that $\det A(t)$ nowhere vanishes on L.

D. A. Kveselava (RŽMat **1957**, no. 5497).

Morita, Kiiti. On the kernel functions for symmetric domains. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 190–212.

This paper is very rich in results. Besides some general properties of the Bergman kernel function, it is mainly concerned with the "bounded symmetric domains" of the p-dimensional complex euclidean space C^p (as defined by E. Cartan [Abh. Math. Sem. Hamburg. Univ. **11** (1935), 116–162]; they are bounded domains $D \in C^p$ that possess a group Ω of analytic homeomorphisms which contains mappings taking any given point in D into any other given point in D, and for each point P in D the group Ω contains an involution with the single fixed point P). There are four main types of "irreducible bounded symmetric domains" such that any bounded symmetric domain can be represented as a direct product of these. The irreducible symmetric domains are characterized by matrix conditions.

The author derives an explicit formula for the Bergman kernel function of these irreducible symmetric domains in terms of the characterizing matrices. As the Bergman kernel function of a product domain is the product of the kernel functions of the domains (for which the author gives a new proof) the kernel function of any bounded symmetric domain is therewith determined explicitly.

For one type of the irreducible symmetric domain "$\mathfrak{M}_{(n)}$" an explicit formula for all its automorphisms is given. A special norm is introduced which leads to an interesting generalization of Schwarz's lemma for $\mathfrak{M}_{(n)}$. — For any bounded irreducible symmetric domain, the Bergman metric is, up to a constant factor, the only invariant Hermitian Kähler metric, and it is calculated explicitly.

The function

$$H_D(x, y)=v(D)^q\left(\frac{K_D(x, \bar{y})K_D(y, \bar{x})}{K_D(x, \bar{x})}\right)^q$$

(where $v(D)$ is the euclidean volume), is for a certain q harmonic, and

$$f(x)=\int_{\mathfrak{B}_0(D)} H_D(x, y)f(y)d\mu_y,$$

holds for all $f \in \mathfrak{L}^2(D)$, where $\mathfrak{B}_0(D)$ is a certain subset of the boundary of D. This formula is connected with a different integral formula of S. Bochner [Ann. of Math. (2) **45** (1944), 686–707; MR **6**, 123].

For general domains D the following is shown: $d(y_1,y_2)=[K_D(y_1, \bar{y}_1)+K_D(y_2, \bar{y}_2)-K_D(y_1, \bar{y}_2)-K_D(y_2, \bar{y}_1)]^{\frac{1}{2}}$ is a metric in D that defines the topology of the C^p in D. Also, for the "bounded analytically homogeneous domains" a generalization of Schwarz's lemma as well as a distortion theorem is obtained.

Finally, it should be noted that the result of the author, "if D is a Cartan circular domain, then $K_D(0, \bar{z})=v(D)^{-1}$ ($v(D)$ the euclidean volume of D)", together with a recent result of Maschler [Pacific J. Math. **6** (1956), 501–516; MR **18**, 473] implies that the Cartan circular domains are Bergman minimal domains. *H. J. Bremermann.*

See also: Approximations, Orthogonal Functions: Rémès; Tumarkin. **Partial Differential Equations:** Mihailov; Danilyuk; Šabat; Landis. **Integral and Integrodifferential Equations:** Hvedelidze. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Bremermann. **Convex Domains, Integral Geometry:** Huber. **Complex Manifolds:** Séminaire Cartan. **Numerical Methods:** Salzer.

Geometrical Analysis

Král, Josef; und Mařík, Jan. Der Greensche Satz. Czechoslovak Math. J. **7(82)** (1957), 235–247. (Russian summary)

In diesem Artikel wird unter gewissen Voraussetzungen über die Funktionen P, Q die Formel

$$\int_K P\,dx+Q\,dy=\int_G\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cdot \operatorname{ind}_K\,dxdy$$

bewiesen, wo K eine beliebige geschlossene rektifizierbare Kurve, $\operatorname{ind}_K z$ den Index des Punktes z in bezug auf K und G die Menge $E[z; \operatorname{ind}_K z \neq 0]$ bedeutet.

Zusammenfassung des Autors.

See also: Differential Geometry: Lagally. **Manifolds, Connections:** Duff; Nakae. **Relativity:** Buchdahl.

Harmonic Functions, Convex Functions

Gehring, F. W. The Fatou theorem and its converse. Trans. Amer. Math. Soc. **85** (1957), 106–121.

The author further extends both the Fatou theorem and its converse, the latter by means of a new two-dimensional form of the Wiener Tauberian theorem. The results are complicated and we state only one theorem. Let $u(z)$ be a positive harmonic function in the upper half-plane and let $\alpha(t)$ be the boundary function in the Poisson-Stieltjes integral representation of $u(z)$. Suppose that $u(re^{id})\sim Ar^\delta$ and $D^n_d u(re^{id})\sim Br^{\delta-n}$ as $r\to 0+$, where D^n_d is the nth directional derivative in the direction of the angle d. Then $\alpha(t)\sim Ct^{1+\delta}$ and $-\alpha(-t)\sim Dt^{1+\delta}$ as $t\to 0+$, where the constants C and D are explicitly given in terms of a, b, δ, d, n, A and B. These constants are subject to the conditions: $0<a, b<\pi$; $-1<\delta<1$; and $(b-a)\delta+(d-nb)\not\equiv 0 \bmod \pi$. *L. H. Loomis.*

Ohtsuka, Makoto. Capacité d'ensembles de Cantor généralisés. Nagoya Math. J. **11** (1957), 151–160.

Let $k_1, k_2, \cdots$ be a sequence of integers each >1, and $p_1, p_2, \cdots$ be a sequence of real numbers likewise each >1. Put $l_q=1/k_q p_q$, and let I denote an interval of length $d>0$. The deletion from I of (k_q-1) intervals of equal length so that there remain k_q intervals of equal length $l_q d$ is called the q-operation applied to I. Commence by applying the 1-operation to $[0, 1]$, then apply the 2-operation to each of the intervals $I_{1\nu}$ $(1<\nu<k_1)$ which remain, the 3-operation to each of the remaining intervals $I_{2\lambda}$ $(1<\lambda<k_1k_2)$, and so on. The remaining limit set is

called the generalized Cantor set in E_1 and is denoted by $F=F(k_q, p_q)$. The author's definition in $E_n=\{(x_1, x_2, \cdots, x_n)\}$, euclidean space of n dimensions ($n>2$), is as follows. Let $F_j=F(k_q^{(j)}, p_q^{(j)})$ be a generalized Cantor set on the x_j axis; the product set $F=F_1\times F_2\times\cdots\times F_n$ is called a generalized Cantor set in E_n ($n>2$). It is said to be symmetric if $F_1=F_2=\cdots=F_n$. Theorem: Given $F=F(k_q, p_q)\times\cdots\times F(k_q, p_q)$, a symmetric generalized Cantor set in E_n ($n>1$); in order that it be of α-capacity zero ($0\leqq\alpha<n$), it is necessary and sufficient that

$$\sum_{q=1}^{\infty}\frac{(p_1\cdots p_q)^{\alpha}}{(k_1\cdots k_q)^{n-\alpha}}=\infty\quad(0<\alpha<n)$$

or

$$\sum_{q=1}^{\infty}\frac{\log p_q}{(k_1\cdots k_q)^n}=\infty\quad(\alpha=0).$$

W. W. S. Claytor (Washington, D.C.).

See also: Partial Differential Equations: Slobodeckiĭ. **Differential Geometry:** Lagally. **Manifolds, Connections:** Elianu.

Special Functions

Al-Salam, Waleed A. On the Bessel polynomials. Boll. Un. Mat. Ital. (3) **12** (1957), 227–229.

It is well known that, if the order is half an odd integer, Hankel functions reduce to elementary expressions containing a polynomial in $x=i/z$. In the present note these polynomials are called Bessel polynomials and a formula is derived for the product of two consecutive polynomials. It is shown that this relation may be used for characterizing the set. *D. J. Hofsommer.*

Al-Salam, Waleed A. On a generalized Hermite polynomial. Boll. Un. Mat. Ital. (3) **12** (1957), 241–246.

Toscano [Riv. Mat. Univ. Parma **6** (1955), 117–140; MR **17**, 1082] has studied the polynomial $G_{n,\nu}(x)$ defined by

$$G_{n,\nu}=xG_{n-1,\nu}-(n+\nu)G_{n-2,\nu},\quad G_{0,\nu}=1,\quad G_{1,\nu}=x;$$

for $\nu=-1$, $G_{n,\nu}$ reduces to the Hermite polynomial, while for $\nu=0$, it reduces to the polynomial associated with the Hermite polynomial. This generalization had been introduced by Palamà [ibid. **4** (1953), 363–386; MR **16**, 470].

The present paper contains some additional results concerning the $G_{n,\nu}$. In particular it is proved that

$$H_m(x)G_{n,\nu}(x)=\sum_{r=0}^{m}r!\binom{m}{r}\binom{n+\nu+1}{r}G_{n+m-2r,\nu}(x)\qquad(m\leqq n),$$

where H_m is the Hermite polynomial;

$$G_{n+1,\nu}'(x)G_{n,\nu}(x)-G_{n+1,\nu}(x)G_{n,\nu}'(x)=\sum_{r=0}^{n}r!\binom{n+\nu+1}{r}G_{n-r,\nu}^2(x);$$

$$(G_{n,\nu}'(x))^2-G_{n+1,\nu}'(x)G_{n-1,\nu}'(x)>G_{n,\nu}^2(x)-G_{n+1,\nu}(x)G_{n-1,\nu}(x)\quad(n\geqq1);$$

$$(G_{n,\nu}''(x))^2-G_{n+1,\nu}''(x)G_{n-1,\nu}''(x)>0\qquad(n\geqq2).$$

L. Carlitz (Durham, N.C.).

Mitrinovitch, Dragoslav S. Inégalités pour les dérivées des polynômes de Legendre. Boll. Un. Mat. Ital. (3) **11** (1956), 172–177.

By differentiating the identity

$$(1-2xt+t^2)^{\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n$$

r times, the author obtains simply and elegantly the estimate $|P_n^{(r)}(x)|\leqq|P_n^{(r)}(1)|=(n+r)!/2^r r!(n-r)!$, for $|x|\leqq1$. However neither the result nor the method is new. [Cf. Grosswald, Proc. Amer. Math. Soc. **1** (1950), 553–554; MR **12**, 178; and du Plessis, ibid. **2** (1951), 950; MR **13**, 553.] *G. Klein* (Chicago, Ill.).

Richmond, D. E. Complex numbers and trigonometry. Amer. Math. Monthly **64** (1957), 478–485.

A pedagogical approach to the trigonometric functions via complex numbers.

Myrberg, P. J. Über automorphe Thetafunktionen. S.-B. Berlin. Math. Ges. **1954/55-1955/56**, 30–31.

Summary of a lecture of June, 1955; cf. Myrberg, Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 118–126 [MR **19**, 263].

Gatteschi, Luigi. Sul comportamento asintotico delle funzioni di Bessel di prima specie di ordine ed argomento quasi uguali. Ann. Mat. Pura Appl. (4) **43** (1957), 97–117.

Continuing earlier work [same Ann. (4) **38** (1955), 267–280; MR **17**, 34], the author obtains asymptotic approximations for $J_\nu(x)$ when ν is large and x is near ν. He gives both first approximations in terms of $J_\nu(\nu)$ and Airy functions, and second approximations in terms of Airy functions alone. A valuable feature of his work is a determination of bounds for the error. The functions he approximates are

$$J_\nu(\nu\exp[6^{-1/3}\nu^{-2/3}t])\quad\text{for } t<6^{1/3}\nu^{2/3},$$

and a typical result states that the above function is equal to

$$\Gamma(\tfrac{1}{3})\,(t/3)^{1/2}J_\nu(\nu)\{J_{-1/3}[2(t/3)^{3/2}]+J_{1/3}[2(t/3)^{3/2}]\}+\rho$$

for $0<t<6^{1/3}\nu^{2/3}$, where $\pi\nu|\rho|<t^4+5.6t$ for $\nu\geqq6$. Two subsidiary results are worth mentioning. One is an asymptotic approximation with remainder for $J_\nu'(\nu)$; and the other, the inequality $\pi x^{1/2}\nu^{1/3}|J_\nu(\nu x)|<4$, valid for $x>1$, $\nu\geqq6$. *A. Erdélyi* (Pasadena, Calif.).

Gatteschi, Luigi. Sulle serie inviluppanti e loro applicazioni alla valutazione asintotica delle funzioni di Bessel. Confer. Sem. Mat. Univ. Bari no. **22** (1957), 12 pp.

In this lecture the author recalls van der Corput's theory of enveloping series and then proceeds to obtain an asymptotic expansion for $J_\nu(\nu)$ for large ν in terms of incomplete gamma functions, together with a majorant series for the asymptotic expansion. From the first term of this expansion he obtains

$$J_\nu(\nu)=2^{-2/3}3^{-1/6}\pi^{-1}\nu^{-1/3}\Gamma(\tfrac{1}{3})-\theta\nu^{-5/3},$$

where $0\leqq\theta<1$, for $\nu\geqq6$. *A. Erdélyi.*

Livingston, Donald. Sur une identité opérationnelle et la fonction hypergéométrique. C. R. Acad. Sci. Paris **244** (1957), 3007–3009.

The author proves that if two variables, z and t, are so connected that $(d/dz)^n\phi_n(z)=\psi_n(t)(d/dt)^n$ for some $n>1$ and some $\phi_n(z)$, $\psi_n(z)$, then z and t are connected by a homographic substitution, the relation holds for all positive integers n, and ϕ_n, ψ_n are powers of linear functions of their variables (or constants). He applies this result to differentiation formulas for hypergeometric functions. *A. Erdélyi* (Pasadena, Calif.).

See also: Complex Manifolds: Séminaire Cartan.

Sequences, Series, Summability

Bellman, Richard. The expansions of some infinite products. Duke Math. J. **24** (1957), 353–356.

The expansion of the double products

$$\prod_{r,s=0}^{\infty}(1-x^ry^st),\quad \prod_{r,s=0}^{\infty}(1-x^ry^st)^{-1}$$

has been discussed (independently) by the reviewer [Proc. Amer. Math. Soc. **7** (1956), 558–564; MR **19**, 29] and E. M. Wright [ibid. **7** (1956), 880–890; MR **18**, 793]. In the present paper it is shown first that

$$\prod_{k=1}^{\infty}\frac{1-x^k}{1-x^kz}=\sum_{r=1}^{\infty}\frac{\prod_{k=1}^{r-1}(1-x^{-k})^{-1}}{1-x^rz},$$

and secondly that

$$(*)\qquad \prod_{k,l=1}^{\infty}\frac{1-x^ky^l}{1-x^ky^lz}=\sum_{r,s=1}^{\infty}\frac{b_{rs}}{1-x^ry^sz},$$

where

$$b_{rs}=\prod_{k,l=1}^{r-1,s-1}(1-x^{-k}y^{-l})^{-1}\prod_{k=1}^{\infty}\prod_{l=0}^{s-1}(1-x^ky^{-l})^{-1}\times\prod_{k=0}^{r-1}\prod_{l=1}^{\infty}(1-x^{-k}y^l).$$

The author remarks that a rigorous proof of (*) requires a measure of the irrationality of $\log x/\log y$.

L. Carlitz (Durham, N.C.).

Krzyż, Jan. An inequality concerning series with decreasing positive terms. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 187–193 (1957). (Polish and Russian summaries)

Let $U_n=u_0+u_1+\cdots+u_n$ ($u_0>0$; $u_k\geqq 0$, $k=1, 2, \cdots$). It is shown that if $\{U_n\}$ converges, then

$$\tau_n=\frac{u_0}{U_n}+\frac{u_1}{U_{n-1}}+\cdots+\frac{u_n}{U_0}\to 1.$$

The author gives an example of a divergent sequence $\{U_n\}$ with $u_n\downarrow 0$ for which $\{\tau_n\}$ diverges and $\tau_n>(\frac{1}{2}-\varepsilon)\log n$ ($\varepsilon>0$ arbitrary) for sufficiently large n. He then shows that for any increasing and concave sequence $\{U_n\}$ ($U_0>0$)

$$\tau_n\leqq 1+\tfrac{1}{2}+\cdots+\frac{1}{n+1}=O(\log n).$$

This result implies a result for power series, viz., for $0<r<1$,

$$(U_0+U_1r+U_2r^2+\cdots)\left(\frac{1}{U_0}+\frac{1}{U_1}r+\frac{1}{U_2}r^2+\cdots\right)\leqq\frac{1}{r(1-r)^2}\log\frac{1}{1-r}$$

for increasing concave $\{U_n\}$ with positive terms. Equality holds for $U_n=n+1$ only. *J. G. Herriot.*

Milošević, K. Sur une série géometrique généralisée. Bull. Soc. Math. Phys. Macédoine **7** (1956), 44–50. (Serbo-Croatian. French summary)

La série des puissances $\sum_{k=1}^{\infty}(a+(k-1)d)^nx^k$, $|x|<1$, n entier, peut s'écrir sous la forme $x(1-x)^{-(n+1)}P_n(x)$, où $P_n(x)$ sont des polynômes en x. Pour les coefficients de ces polynômes on donne une expression explicite.

M. Tomić (Belgrade).

Hanna, J. Ray. The Dirichlet series transformation. Amer. Math. Monthly **64** (1957), 576–581.

The author discusses the Dirichlet series transform $f(s)=\sum_{t=0}^{\infty}a^{-st}F(t)$ ($a>1$), where s is complex. He obtains formulas for $F(t)$ in terms of $f(t)$, derives a formula for $f(s+\sigma)$, discusses convolutions, gives some applications and a short table of transforms. The paper is in the nature of a supplement to a paper by T. Fort [same Monthly **62** (1955), 641–645; MR **17**, 860]. *T. Fort.*

Wilansky, Albert. On the Cauchy criterion for the convergence of an infinite series. Amer. Math. Monthly **64** (1957), 469–471.

The author states the familiar Cauchy condition for the convergence of an infinite sequence in matrix form, thus defining a "Cauchy matrix". Let a sequence $x=\{x_n\}$ be said to satisfy a "Cauchy condition" if Ax is null for some Cauchy matrix A. These statements are amplified in the paper. The author is then concerned with the history and proof of the following theorem: No countable collection of Cauchy conditions is sufficient for convergence; more precisely, given a countable collection of Cauchy conditions, there exists an unbounded sequence which satisfied them all. *T. Fort.*

Obrechkoff, Nikola. Sur la sommation absolue des séries lacunaires. C. R. Acad. Bulgare Sci. **8** (1955), no. 4, 1–4. (Russian summary)

Eyraud, Henri. Séries monotones régulièrement divergentes. Ann. Univ. Lyon. Sect. A. (3) **19** (1956), 47–51.

If $\{c_n\}$ and $\{d_n\}$ are both non-increasing sequences of positive terms that tend to 0 as $n\to\infty$, and if $\sum c_n$ is convergent and $\sum d_n$ is divergent, and if finally $c_n\geqq d_n$ for an infinity of values of n, then $\sum d_n$ is said to be 'semi-divergent'. If for a given sequence $\{d_n\}$ no such sequence $\{c_n\}$ exists, then $\sum d_n$ is said to be 'totally divergent'. It is proved that $\sum d_n$ is totally divergent if and only if the lower limit of nd_n is strictly positive. A totally divergent series has 'regular divergence' if $nd_n\to k$ as $n\to\infty$, where $k>0$. If an event is 'successful' n times in $f(n)$ trials, and if $p_n=n/f(n)\to p$ as $n\to\infty$, then $\sum 1/f(n)$ is totally divergent and has regular divergence. The author calls p the 'probability of the event' although no notion of randomness is involved. The terminology would be more justified if the discussion were extended to cover 'almost all' series $\sum 1/f(n)$. *I. J. Good* (Cheltenham).

Agnew, Ralph Palmer. Densities of sets of integers and transforms of sequences of zeros and ones. Trans. Amer. Math. Soc. **85** (1957), 369–389.

Let $(\lambda_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers and let $S(t)=tD(t)$ denote the number of λ_n not exceeding t. The upper density of the sequence is defined by $\bar{D}=\limsup_{x\to\infty}D(x)$; the upper mean density by $\bar{D}_M=\limsup_{x\to\infty}(1/x)\int_0^x D(t)dt$; the upper Poisson density by $\bar{D}_P=\limsup_{x\to\infty}(2/\pi)\int_0^x xD(t)(t^2+x^2)^{-1}\,dt$. These and other upper densities play an important role in various investigations on infinite series and in the theory of functions. The author remarks that $\bar{D}_M$, $\bar{D}_P$ and other upper densities (he is mostly concerned with 8) are of the form $\bar{D}_\phi=\limsup_{x\to\infty}(1/x)\int_0^\infty\phi(t)\,dS(xt)$, where $\phi(t)\geqq 0$, continuous, monotone decreasing and $\int_0^\infty\phi(t)dt=1$. The author's main results are the inequality $\bar{D}_\phi\geqq\sup_{x>0}\int_{(1-\bar{D})x}^x\phi(t)\,dt$ and the assertion that it is best possible. The

second part is achieved by constructing sequences with suitably placed large blocks and gaps. In the special case $\bar{D}_M$ the result is due to the reviewer [C. R. Acad. Sci. Paris **225** (1947), 481–483; MR **9**, 140]. The paper contains also numerical studies of 8 upper densities for the particular sequences studied by the author. It concludes with fragmentary results on the validity of $\bar{D}_1 \leqq \bar{D}_2$, where $\bar{D}_1$ and $\bar{D}_2$ are any of the 8 above mentioned upper densities. *A. Dvoretzky* (Princeton, N.J.).

Jurkat, W. B. Über die Umkehrung des Abelschen Stetigkeitssatzes mit funktionentheoretischen Methoden. Math. Z. **67** (1957), 211–222.

In this paper the author shows systematically how his function theoretic methods [Arch. Math. **7** (1956), 122–125; MR **18**, 31] lead to simple and elegant proofs of Tauberian theorems for Abel summability. In some respects his results are more general than those of Hardy and Littlewood. The most noteworthy example is the following. Let $\sum a_n$ be Abel summable to s; that is, $f(z)=\sum a_n z^n$ converges for $|z|<1$ and $f(x)\to s$ as $x \uparrow 1$. Littlewood's theorem asserts that the Tauberian condition $na_n=O(1)$ implies the convergence of $\sum a_n$. The author shows that the weaker Tauberian condition $\sum_{j\leq n} |ja_j|^2=O(n)$ already implies the convergence of $\sum a_n$, and (therefore) its (C, k) summability for every $k>-\frac{1}{2}$. The proof depends on the equivalence of the Tauberian condition with the estimate $\int_{-\pi}^{\pi} |zf'(z)|^2 d\varphi = O(1)/(1-r)$ $(z=re^{i\varphi})$. Adroit use of Cauchy's formula for the partial sums and the $(C, k+1)$ means of the a_n in terms of $f(z)/(1-z)$ and $f(z)/(1-z)^{k+1}$, and an application of Montel's theorem, now lead to the result in a few lines.

The following are some of the other 17 theorems. Let $s_n=\sum_{j\leq n} a_j$, $\sum a_n z^n=f(z)$, $\sum s_n z^n=F(z)$, $p>1$ and $(1/p)+(1/q)=1$. (i) Under each of the conditions

$$\sum_{j\leq n} |s_j|^p=O(n), \int_{-\pi}^{\pi} |F(z)|^q d\varphi=O(1)/(1-r)^{q-1}$$

(equivalent only if $p=q=2$), one has $(A)\Rightarrow(C, k)$ for $k>1/p$, but not for $k=1/p$. (ii) Under each of the conditions

$$\sum_{j\leq n} |ja_j|^p=O(n), \int_{-\pi}^{\pi} |zf'(z)|^q d\varphi=O(1)/(1-r)^{q-1}$$

one has $(A)\Rightarrow(C, k)$ for $k>(1/p)-1$, but not for $k=(1/p)-1$. Theorems dealing with the limit cases $p=1$ and $p=\infty$ are proved separately. In another class of theorems the Tauberian condition involves (C, α) means of the s_n or the na_n.

{Unlike the Karamata-Wielandt real variable proof [see Wielandt, Math. Z. **56** (1952), 206–207; MR **14**, 265] of Littlewood's theorem which depends on approximation by polynomials the present method is not easily adapted to the case of one-sided Tauberian conditions. It can also be stated that the author's method is not suited for estimates of the remainder in Littlewood's theorem as given by Freud [Acta Math. Acad. Sci. Hungar. **2** (1951), 299–308; MR **14**, 361; and subsequent papers] and the reviewer [Duke Math. J. **18** (1951), 723–724; Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 281–293; MR **13**, 227; **15**, 119; and subsequent papers]. The reason is that Tauberian conditions such as $\sum_{j\leq n} |ja_j|^2=O(n)$ do not imply such sharp estimates for the partial sums s_n as one has under the conditions $na_n=O(1)$ or $na_n>-K$.} *J. Korevaar* (Madison, Wis.).

Jakimovski, Amnon. Some Tauberian theorems. Pacific J. Math. **7** (1957), 943–954.

Rajagopal, C. T. Simplified proofs of "Some Tauberian theorems" of Jakimovski. Pacific J. Math. **7** (1957), 955–960.

The authors consider the "Hölder" limitation method defined by the following Hausdorff transformation. The (H, α) transform of the sequence $\{s_n\}$ is the sequence $\{t_n\}$ given by the condition that $\Delta^n t_0=(n+1)^{-\alpha}\Delta^n s_0$ $(n=0, 1, 2, \cdots)$. For $\alpha=-k$ $(k=1, 2, \cdots)$ this limitation method is weaker than convergence. The principal result obtained by the authors is that a sequence $\{s_n\}$ is $(H, -k)$ limitable if and only if $\{s_n\}$ is Abel limitable and $n^k\Delta^k s_n\to 0$ as $n\to\infty$. In another result $\{s_n\}$ is supposed to be Abel limitable and the sequence $\binom{n}{k}\Delta^k s_{n-k}$ $(n=0, 1, 2, \cdots)$ is supposed to be $(H, \alpha+k)$ limitable to zero (or $(H, \alpha+k)$ bounded from below). The conclusion is that $\{s_n\}$ is limitable by the method (H, α) (or by the method $(H, \alpha+1)$, respectively). *J. Korevaar* (Madison, Wis.).

See also: Elementary Algebra: Zeckendorf. **Analytic Theory of Numbers:** Selberg. **Approximations, Orthogonal Functions:** Ghizzetti. **Integral Transforms:** Devinatz.

Approximations, Orthogonal Functions

Balázs, J.; and Turán, P. Notes on interpolation. II. Explicit formulae. Acta Math. Acad. Sci. Hungar. **8** (1957), 201–215.

In Part I [Surányi and Turán, same Acta **6** (1955), 67–80; MR **17**, 148] the following problem was considered: is there a polynomial R_n of degree at most $2n-1$ for which $R_n(\xi_\nu)=b_\nu$ and $R_n''(\xi_\nu)=\beta_\nu$ (ξ_ν are points in $[-1, 1]$)? Here it is supposed that the ξ's are the zeros of $(1-x^2)P_{n-1}'(x)$ (P_n are the Legendre polynomials), and that $n=2k$. In this case the authors can determine the unique R_n explicitly; the formulas are rather complicated and are presented in several versions. *R. P. Boas, Jr.* (Evanston, Ill.).

Rémès, E. J. Sur les points d'écart fixes des solutions des problèmes d'approximation de Tchebyscheff aux paramètres entrant linéairement. I. Ukrain. Mat. Ž. **9** (1957), 44–65. (Russian. French summary)

For a finite subset A of complex $(n+1)$-space and for $z=(z_1, \cdots, z_n)$, let $\rho(z)=\max|\sum a_k z_k - a_0|$, $a\in A$, and $A(z)$ be the part of A on which $\rho(z)$ is attained. Let Z be the convex set [e.g., Remez, same Ž. **5** (1953), 3–49; MR **15**, 407] of all z with minimal $\rho(z)$, Z^* its relative interior. The points in the intersection A_0 of all $A(z)$, z in Z, are called fixed deviation points. Then (1) $A(z)=A_0$ for z in Z^*; (2) $A(z)=A_0$ for z in $Z-Z^*$; (3) the real and the complex linear spaces spanned by Z coincide. For infinite A, (1) persists, (2) and (3) do not. {Reviewer's remark: (1) and (2) correspond to well-known properties of the lowest face of a convex polyhedron.} *T. S. Motzkin* (Los Angeles, Calif.).

Morozov, M. I. Extension of two theorems of S. N. Bernštein to analytic functions of several variables. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki. **10** (1956), 6–18. (Russian)

The theorems of Bernstein mentioned in the title characterize functions which are analytic in an ellipse

(or periodic and analytic in a strip) in terms of the asymptotics of their best approximation by polynomials (or by trigonometric polynomials). The author extends these theorems to functions of several variables. Let

$$\pi_n(x_1, \cdots, x_k)=\sum_{\substack{0\le h_i\le n\\ 1\le i\le k}} a_{h_1\cdots h_k}\, x_1^{h_1}\cdots x_k^{h_k}$$

designate a polynomial of degree $\le n$ in each of its k variables x_i. For functions $f(x_1, \cdots, x_k)$ which are continuous in the hypercube $-1\le x_i\le 1$ $(i=1, \cdots, k)$, set

$$E_n[f]=\min_{\pi_n}\max_{-1\le x_i\le 1}|f(x_1, \cdots, x_k)-\pi_n(x_1, \cdots, x_k)|.$$

Theorem. In order for the function $f(x_1, \cdots, x_k)$ to be analytic in the k-dimensional cube $-1\le x_i\le 1$, it is necessary and sufficient that $\limsup_{n\to\infty}[E_n(f(x_1, \cdots, x))]^{1/n}=q<1$, where $q\ge 0$ and $1/R_0\le q\le 1/R^*$. Here R_0 and R^* are the sums of the semi-axes of certain ellipses derived from the domain of regularity of f.

A related theorem for periodic analytic functions of many variables is also given, as are certain modifications of these theorems. *P. Davis* (Washington, D.C.).

Crum, M. M. Corrigendum and addendum: On the theorems of Müntz and Szász. J. London Math. Soc. **32** (1957), 512.

The paper in question appeared in the same J. **31** (1956), 433–437 [MR **18**, 207]. The addendum is a proof of the fact that (with notation as in the cited review) (B′) does not imply (A) when X is L^p and $1<p<2$.

R. P. Boas, Jr. (Evanston, Ill.).

Ghizzetti, Aldo. Sui coefficienti di Legendre-Stieltjes di una funzione non decrescente. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 753–758.

The author gives necessary and sufficient conditions for a set of numbers to be Legendre-Stieltjes coefficients of a non-decreasing function, using a method similar to the one he used for Fourier-Stieltjes coefficients [same Rend. (8) **20** (1956), 580–583; MR **18**, 735]. *R. P. Boas, Jr.*

Tumarkin, G. C. Approximation in the mean to functions on rectifiable curves. Mat. Sb. N. S. **42** (**84**) (1957), 79–128. (Russian)

A self-contained paper discussing approximation of a function $f(t)$ $(0\le t\le 2\pi)$ by linear combinations of e^{int} $(n=0, 1, 2, \cdots)$ in the norm of the space $L^p(d\sigma; 0, 2\pi)$, $p>0$, given by $\|f\|=\int_0^{2\pi}|f(t)|^p d\sigma(t)$, where σ is a bounded increasing function. More generally, the problem consists in approximation of a function $f(z)$ defined on a closed rectifiable Jordan curve γ by polynomials in the norm of $L^p(d\sigma; \gamma)$. Compared with the summary of the author's results [Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 21–24; MR **14**, 154], the following theorem (formulated here for the interval) is new. If $\int_0^{2\pi}\log\sigma'(t)dt=-\infty$, then for each $f\in L^p(d\sigma; 0, 2\pi)$ and each $\Phi(z)$ analytic in $|z|<1$ there is a sequence of polynomials $\Pi^k(z)$ with the properties: $\Pi^k(z)\to\Phi(z)$ uniformly on compact subsets of $|z|<1$ and $\|f-\Pi^k\|\to 0$. *G. G. Lorentz* (Detroit, Mich.).

Ganelius, Tord. On one-sided approximation by trigonometrical polynomials. Math. Scand. **4** (1956), 247–258.

Soit $f(q)$ une fonction réelle bornée de période 2π; on considère des polynomes trigonométriques S_n et s_n, de degré $(n-1)$ au plus, et tels que: $S_n\ge f\ge s_n$, et on se propose, pour n donné, d'étudier, dans certaines classes, le minimum de $\|S_n-f\|=\pi^{-1}\int_0^{2\pi}(S_n-f)d\theta$ et de $\|f-s_n\|=\pi^{-1}\int_0^{2\pi}(f-s_n)d\theta$, et d'autres questions analogues. Pour

$$b_m=-2m!\sum_1^\infty k^{-m}\cos(k\theta-\tfrac{1}{2}m\pi),$$

avec $m\ge 1$, on détermine les polynomes minimants $T_{m,n}$ et $t_{m,n}$, et les meilleures constantes:

$$\|T_{m,n}-b_m\|=n^{-m}\sup b_m;\ \|b_m-t_{m,n}\|=-n^{-m}\inf b_m.$$

Cela permet de montrer que, pour f telle que $f^{(r-1)}$ est à variation bornée, et de variation V_r dans $[0, 2\pi]$, il y a un S_n tel que: $\|S_n-f\|\le C_rV_r/n^{r+1}$ avec la meilleure constante:

$$C_r=\frac{\sup b_{r+1}-\inf b_{r+1}}{4\pi(r+1)!}.$$

On précise ainsi un résultat de Freud [Acta. Sci. Math. Szeged **16** (1955), 12–28; MR **17**, 30]. Plus généralement, on peut trouver un polynôme W_n tel que:

$$W_n\ge f,\ \|W_n-f\|\le\frac{D_rV_r}{n^{r+1}},\ \operatorname{var}(W_n-f)\le\frac{D_rV_r}{n^r},$$

où D_r est une constante. On généralise ensuite un résultat de Hörmander [Math. Scand. **2** (1954), 33–45; MR **16**, 354]. *J. Favard* (Paris).

Chen, Kien-Kwong. On the series of orthogonal polynomials. Sci. Record (N.S.) **1** (1957), no. 2, 13–18.

Let $\{P_n(x)\}$ be a system of polynomials orthonormal with respect to a positive weight function $\tau(x)\in L$ over $\langle a, b\rangle$; and let $f\in L$, with modulus of continuity $\omega=\omega(\delta, f)$, have Fourier coefficients c_n with respect to this system. The following results are proved amongst others. I. If $\omega(\delta, f)<[L_1(1/\delta)\sqrt{(L_2(1/\delta)\cdots L_{j-1}(1/\delta)L_j^{1+\varepsilon}(1/\delta))}]^{-1}$ for some $\varepsilon>0$ and integral j, then the Fourier series of f converges almost everywhere [for $j=2$, proved by Ulianov, Mat. Sb. N.S. **40**(**82**) (1956), 95–100; MR **18**, 803; $L_j(x)$ denotes the j-times iterated logarithm]. II. If

$$\sqrt{((x-a)(b-x))}\tau(x)\le A,$$

and f is continuous, then $\sum|c_n|^p<\infty$ for

$$p>\limsup 1/(\tfrac{1}{2}\log n-\log\omega(1/n)).$$

If $f\in\operatorname{Lip}\alpha$, this gives $p>2/(1+2\alpha)$ [O. Szász, S.-B. Math.-Phys. Kl. Bayer. Akad. Wiss. **1922**, 135–150]. III. Under the same condition on $\tau(x)$, if $f(x)$ is continuous and of bounded variation, then $\sum|c_n|^p<\infty$ for

$$p>\limsup\log n/(\log n-\tfrac{1}{2}\log\omega(1/n));$$

and if $\omega(\delta)<[L_1(1/\delta)L_2(1/\delta)\cdots L_{j-1}(1/\delta)L_j^{1+\varepsilon}(1/\delta)]^{-2}$, then $\sum|c_n|<\infty$. *W. W. Rogosinski* (Boulder, Colo.).

Alexits, G.; und Králik, D. Über die Bedeutung der strukturellen Eigenschaften einer Funktion für die Konvergenz ihrer Orthogonalentwicklungen. Acta Sci. Math. Szeged **18** (1957), 131–139.

Let $\{\varphi_n(x)\}$ be an orthonormal system over $\langle a, b\rangle$, and $f\in L^2$ have Fourier coefficients c_n with respect to it. It is further assumed that the system has the property (satisfied, for instance, by the trigonometrical system):

$$\|f(x)-\sum_0^{n-1}c_k\varphi_k(x)\|_2=[\sum_n^\infty c_k^2]^{\frac{1}{2}}=O(\omega_2(1/n, f)),$$

where $\omega_2(\delta, f)$ is the quadratic modulus of continuity of f over $\langle a, b\rangle$. Let $\lambda(x)$, for large $x>0$, be non-negative, in-

creasing, and concave from below, and let $\Phi(x)$ be non-negative and increasing for $x>0$. The main result is that

(*) $$\omega_2(\delta, f)=O(\Phi(1/\delta)^{-1})$$

together with

(**) $$\int_1^\infty \lambda'(x)\Phi(x)^{-1}dx<\infty$$

imply that $\sum_1^\infty c_n^2\lambda(n)<\infty$. The most important case is $\lambda(x)=\log^2 x$, since $\sum_1^\infty c_n^2 \log^2 n<\infty$ implies the convergence of the Fourier series almost everywhere (Rademacher-Menchoff). In the case of the trigonometric system the converse of the above theorem holds: If $\sum_1^\infty c_n^2\lambda(n)<\infty$, then, with $c_n^2=a_n^2+b_n^2$, (*) and (**) hold for a suitable $\Phi(x)$. Here the case $\lambda(x)=\log x$ is important, since $\sum_1^\infty c_n^2 \log n<\infty$ implies the convergence of the Fourier series almost everywhere [Kolmogoroff-Plessner]. There are also results of similar type involving a 'local' modulus $\omega_2(\delta, f)$ of continuity only. [Cf. Alexits, Acta Math. Acad. Sci. Hungar. **4** (1953), 95–101; MR **15**, 619.] *W. W. Rogosinski.*

See also: Numerical Methods: Remez.

Trigonometric Series and Integrals

Slater, N. B. Some formulae of P. Stein and others concerning trigonometrical sums. Proc. Cambridge Philos. Soc. **50** (1954), 33–39.

$G_a(T, \varphi_1, \cdots, \varphi_n)$ sei die Anzahl der Nullstellen von $f(t)-a$ im Intervall $0\leq t<T$. Dabei ist

$$f(t)=\sum_{r=1}^n a_r \cos 2\pi(\nu_r t+\varphi_r),$$

$a_r\neq 0$ und $\nu_r>0$ $(r=1, \cdots, n)$. Es ist $E_a(\varphi_1, \cdots, \varphi_n)=\lim_{T\to\infty} G_a(T, \varphi_1, \cdots, \varphi_n)/T$ die asymptotische Frequenz und $E_a=\bar{G}_a(T)/T$ die mittlere Phasenfrequenz, wobei

$$\bar{G}_a(T)=\int_0^1 \cdots \int_0^1 G_a(T, \varphi_1, \cdots, \varphi_n)d\varphi_1, \cdots, d\varphi_n$$

gilt. Für E_a hat Kac [Amer. J. Math. **65** (1943), 609–615; MR **5**, 96] unter der Voraussetzung der linearen Unabhängigkeit der ν_r eine Darstellung angegeben und bewiesen. Verf. führt eine Untersuchung von Stein [dieselben Proc. **31** (1935), 455–467], die für $n=2$ galt, unter Benutzung von unveröffentlichten Ergebnissen von Stein weiter und beweist Darstellungen von E_a und $E_a(\varphi_1, \cdots, \varphi_n)$, wobei über die ν_r außer $\nu_r>0$ keine weiteren Voraussetzungen gemacht werden. *W. Haacke* (Zbl **57** (1956), 304).

Wintner, Aurel. On the reduction (mod 1) **of completely monotone functions** $(0, \infty)$. Ann. Mat. Pura Appl. (4) **43** (1957), 299–312.

It is assumed that the real function $f(x)$ is completely monotone for $0<x<\infty$ [i.e., $(-1)^k f^{(k)}(x)\geqq 0$ when $k\geqq 0$]; that it is L-integrable over $\langle 0, 1\rangle$; and that $f(x)\to 0$ as $x\to\infty$. These assumptions are equivalent to a representation $f(x)=\int_0^\infty e^{-xs}d\mu(s)$, where $\mu(s)\nearrow$, $\mu(+0)=\mu(0)$, and $\int_1^\infty s^{-1}d\mu(s)<\infty$. The paper studies the associated functions $f^*(x)=\lim_{N\to\infty}\{\sum_{n=0}^{N-1} f(x+n)-\int_0^N f(u)du\}$, $f_1(x)=\frac{1}{2}[f^*(x)+f^*(1-x)]$, and $f_2(x)=\frac{1}{2}[f^*(x)-f^*(1-x)]$, the latter two for $0<x<1$. It is shown that $f_2(x)$ [but, in general, not $f_1(x)$] is completely monotone in $0<x<\frac{1}{2}$, and so is $-f_1'(x)$. Also, on putting $t=2\pi x$, one has, for $0<t<2\pi$, $f_1(t)=\sum_{n=1}^\infty a_n \cos nt$, $f_2(t)=\sum_{n=1}^\infty b_n \sin nt$, where $a_n=\int_0^\infty (s+n^2/s)^{-1}d\mu(s)$ and $b_n=\int_0^\infty (n+s^2/n)^{-1}d\mu(s)$ and the series converge. A particular case is the often studied pair $c_\lambda(t)=\sum_1^\infty n^{-\lambda}\cos nt$ and $s_\lambda(t)=\sum_1^\infty n^{-\lambda}\sin nt$, $0<\lambda<\infty$, when $s_\lambda(t)$ and $-c_\lambda'(t)$ are completely monotone for $0<t<\pi$. [Cf. A. Wintner, Phil. Mag. (7) **38** (1947), 495–504; MR **9**, 235.] *W. W. Rogosinski.*

Konyuškov, A. A. On Lipschitz classes. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 423–448. (Russian)

Given a Fourier series $f(x)\sim a_0+\sum_{n=1}^\infty a_n \cos nx$, the problem of determining the properties of the function determined by the series $\lambda_0 a_0+\sum_{n=1}^\infty \lambda_n a_n \cos nx$ has been intensively studied [see Zygmund, Trigonometrical series, Warszawa-Lwów, 1935]. Hardy initiated the study of the associated series $\sum_{n=1}^\infty A_n \cos nx$, where $A_n=\sum_{k=1}^n a_k/k$, Bellman [Bull. Amer. Math. Soc. **50** (1944), 741–744; MR **6**, 125] studied the corresponding problem for the case where $A_n=\sum_{k=n}^\infty a_k/k$, and Loo Ching-Tsün [Amer. J. Math. **71** (1949), 269–282; MR **10**, 603] studied both aspects. These results were in relation to L^p-classes. The author considers the Lip(a, p) classes and obtains analogues of the foregoing results. *R. Bellman.*

Tomić, M. Remarque sur un procédé de sommation des séries de Fourier. Bull. Soc. Math. Phys. Macédoine **6** (1955), 35–43. (Serbo-Croatian. French summary)

Let $f(x)$ be continuous of period 2π and $s_n(x)$ the nth partial Fourier sum. Let $\omega(t)$ be the oscillation of f on an interval of length t. Let $k(n)$ be odd-integer valued and bounded as a function of n. The author proves that a sufficient condition for uniform convergence, with respect to x, of the series $\frac{1}{2}|s_n(x+\alpha_n)-s_n(x-\alpha_n)|$ to $f(x)$ is that $\alpha_n+\pi k(n)/(2n+1)+1/(n \log n)=o(1/\omega(n))$.

J. M. Danskin (Princeton, N.J.).

Michlin, S. G. Fourier integrals and multiple singular integrals. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 143–155. (Russian. English summary)

The main results were announced in Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 701–703 [MR **18**, 304; lines 11 and 12 of the review should be interchanged].

R. P. Boas, Jr. (Evanston, Ill.).

See also: Approximations, Orthogonal Functions: Tumarkin; Ganelius; Alexits and Králik. **Topological Vector Spaces:** Cotlar.

Integral Transforms

Herz, C. S. A note on the span of translations in L^p. Proc. Amer. Math. Soc. **8** (1957), 724–727.

A function $f\in L\cap L^p$ on $(-\infty, \infty)$ has the Wiener closure property (C) if the linear extension of the set of translates of f is dense in L^p. The Wiener problem concerns itself with finding necessary and sufficient conditions on the zeros of the Fourier transform $\hat{f}$ of f which insure that it has property (C). This problem has been shown to be equivalent in the range $2\leq p<\infty$ to finding conditions on $\hat{f}$ such that certain integral equations, involving f in their kernels, have no non-trivial solutions [H. Pollard, same Proc. **2** (1951), 100–104, MR **13**, 31; I. E. Segal, Trans. Amer. Math. Soc. **61** (1947), 69–105; MR **8**, 438]. In the present paper the author finds conditions of this nature on f for p in the range $1<p<2$ which are implied by (C). *R. E. Fullerton.*

Devinatz, A. On infinitely differentiable positive definite functions. Proc. Amer. Math. Soc. **8** (1957), 3–10.

The author shows that if f is an infinitely differentiable function on an open interval containing the origin then f is the restriction of a positive definite function (i.e., the characteristic function of a distribution function) on $(-\infty, \infty)$ provided that (a) $\{(-i)^k f^{(k)}(0)\}$ is a determined Hamburger moment sequence and (b) for each m there is a positive M_m such that

$$|\sum_{k=0}^{m} \xi_k(-i)^k f^{(k+m)}(x)|^2 \leq M_m \sum_{r,s=0}^{n} \xi_r \bar{\xi}_s (-i)^{r+s} f^{(r+s)}(0)$$

for every $\{\xi_k\}$. On the other hand, (a) is not necessary for f to be positive definite; and (b) together with the fact that $\{(-i)^k f^{(k)}(0)\}$ is a Hamburger moment sequence is not sufficient, although both these conditions are necessary. The proofs use the theory of operators in Hilbert space.

R. P. Boas, Jr. (Evanston, Ill.).

Stein, E. M. Note on singular integrals. Proc. Amer. Math. Soc. **8** (1957), 250–254.

This paper deals with the continuity of singular integral operators

$$\text{P.V.} \int_{E_n} H(x, x-y) f(y) dy$$

such as the ones considered by the reviewer and A. Zygmund [Acta Math. **88** (1952), 85–139; Amer. J. Math. **78** (1956), 289–309; MR **14**, 637; **18**, 894]. The author generalizes known results of Hardy and Littlewood [Duke Math. J. **2** (1936), 354–382] and Babenko [Dokl. Akad. Nauk SSSR (N.S.) **62** (1948), 157–160; MR **10**, 249] concerning the one-dimensional case and proves that these operators are continuous on the space of functions $f(x)$ such that $|f(x)|^p |x|^\beta$ is integrable over E_n, provided that $1<p<\infty$ and $-np^{-1}<\beta<nq^{-1}$, where $p^{-1}+q^{-1}=1$, and $H(x, z)$ is bounded on $|z|=1$.

A. P. Calderón.

See also: Topological Vector Spaces: Cotlar.

Ordinary Differential Equations

★**Sansone, G.; e Conti, R. Equazioni differenziali non lineari.** Edizioni Cremonese, Roma, 1956. xix+647 pp. 7500 Lire.

Nel presente volume sono studiate, con mezzi prevalentemente geometrici, le proprietà asintotiche delle soluzioni dei sistemi di equazioni differenziali ordinarie di tipo normale

$$\text{(A)} \qquad \frac{dx_i}{dt} = f_i(t; x_1, x_2, \cdots, x_n) \qquad (i=1, 2, \cdots, n).$$

Il primo capitolo è dedicato ad una sintetica esposizione dei risultati di carattere generale relativi ai sistemi (A). La parte centrale del volume è invece dedicata al caso particolare dei sistemi

$$\text{(B)} \qquad \frac{dx}{dt} = f(t; x, y), \quad \frac{dy}{dt} = g(t; x, y),$$

che traducono i numerosi problemi suggeriti dalle scienze applicate corrispondenti a sistemi meccanici con un sol grado di libertà. In questa parte del volume trova posto una rielaborazione ed una rassegna, corredata da un'ampia bibliografia che giunge fino al 1955, di molti dei risultati più importanti da quelli classici fino ai più recenti. Ai sistemi (B) autonomi, vale a dire con f e g indipendenti da t, sono dedicati i capitoli II, III, IV, V, VI, secondo il seguente schema: Cap. II. Caso lineare, sistemi omogenei, caso analitico, problema del centro, singolarità all'infinito. Cap. III. Singolarità di Briot-Bouquet. Cap. IV. Teoria generale dello spazio delle fasi: insieme limite, cicli piani, punti singolari isolati e loro classificazione, indice, sistemi di vettori sul cilindro e sul toro. Cap. V. Problema della conservazione di determinate configurazioni nel piano delle fasi sotto l'azione di una perturbazione costante nel tempo e, in particolare, il problema della stabilità di struttura. Nel cap. VI sono prese in esame alcune delle più importanti equazioni della forma $d^2x/dt^2=f(x, dx/dt)$, quali l'equazione del pendolo, l'equazione di van der Pol, l'equazione di Liénard e sue generalizzazioni, un'equazione della dinamica dei fili, ecc. Il cap. VII tratta invece i sistemi (B) non autonomi e costituisce una rassegna coordinata di numerosi risultati interessanti le applicazioni finora pubblicati solo in riviste di matematica e di tecnica. Nei due successivi capitoli viene ripreso lo studio dei sistemi (A) e precisamente nel cap. VIII viene tratteggiata la teoria dei sistemi lineari e nel cap. IX è invece esposta la teoria della stabilità dei sistemi (A) nelle diverse accezioni (secondo metodo di Liapunov, stabilità dei sistemi lineari, stabilità in prima approssimazione, equivalenza asintotica). Molti di questi risultati appaiono per la prima volta in un volume.

Questo volume si inserisce molto degnamente nella Collana delle Monografie del Consiglio Nazionale delle Ricerche italiano, e può essere consultato utilmente sia dai cultori di matematica pura che da quelli di matematica applicata.

G. Stampacchia (Genova).

Pliś, A. One-sided non-uniqueness in ordinary differential equations. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 583–588, XLIX–L. (Russian summary)

The author has succeeded in constructing an example of a system of two differential equations (*) $x_1'=f_1(t, x_1, x_2)$, $x_2'=f_2(t, x_1, x_2)$, where f_1 and f_2 are bounded continuous functions, defined throughout the entire real (t, x_1, x_2)-space R^3, such that each point of R^3 is a point of left-sided uniqueness and also a point of right-sided non-uniqueness for solutions of system (*).

To define right-sided non-uniqueness of solutions let P denote any point (t, x_1, x_2) in R^3 and set

$$c(s; P) = \max[(y_1(s)-z_1(s))^2+(y_2(s)-z_2(s))^2]^{\frac{1}{2}}$$

as the curves $\{y_1(s), y_2(s)\}$ and $\{z_1(s), z_2(s)\}$ range over all possible integral curves of system (*) passing through point P. The point P is called a point of right-sided non-uniqueness for system (*), if for any positive number r there exists a number s, $t<s<t+r$, such that $c(s; P)>0$. Otherwise P is a point of right-sided uniqueness. Left-sided uniqueness and non-uniqueness are analogously defined.

H. L. Turrittin (Minneapolis, Minn.).

Terracini, A. Aspetti proiettivi nella teoria delle equazioni differenziali. Rend. Sem. Mat. Messina **1** (1955), 115–119.

Let (u, v) denote curvilinear coordinates of a point P on a surface S. A set of ∞^2 curves of the cubic type is composed of the integral solutions of a second order differential equation of the form

$$\text{(1)} \qquad v''=A(u, v)+B(u, v)v'+C(u, v)v'^2+D(u, v)v'^3.$$

Examples of such systems are the geodesics of the surface S, natural families, isogonal trajectories, and axial systems. The class of all sets of ∞^2 curves of the cubic type is

topologically invariant, although no two such sets are topologically equivalent, in general. A system of ∞^3 curves of the type (F) consists of the integral solutions of a third order differential equation of the form

(F) $v'''=F(u, v, v')+G(u, v, v')v''+H(u, v, v')v''^2$.

The class of all such sets of the type (F) is topologically invariant. When F is identically zero, a system of ∞^3 curves of the type (F) is said to be of the type (G). The types (G) and (F) arose in the study of dynamical families of ∞^3 curves on a plane and also on a surface S. The type (G) is not topologically invariant but is projectively invariant when (u, v) are considered to be non-homogeneous projective coordinates of a point P on a plane S. The author develops certain interesting projective properties of the systems of curves of the cubic type and the types (F) and (G). *J. De Cicco* (Chicago, Ill.).

Babister, A. W. Stability and response of systems satisfying a second-order linear differential equation with time-dependent coefficients. Aero. Quart. **8** (1957), 78–86.

The differential equation considered is

$$(a_2+b_2\tau)\frac{d^2x}{d\tau^2}+(a_1+b_1\tau)\frac{dx}{d\tau}+(a_0+b_0\tau)x=0,$$

where all the a's and b's are real constants. The nature of the solutions is investigated in the neighbourhood of the singular point $(\tau=-a_2/b_2)$, and the conditions are found for logarithmic terms to be absent. The conditions for stability for large values of τ are determined; the system is stable if $(a_0+b_0\tau)$, $(a_1+b_1\tau)$ and $(a_2+b_2\tau)$ are all positive for large values of τ. The form of the response is considered and its oscillatory (or non-oscillatory) nature investigated. The Sonin-Pólya theorem is used to determine simple inequalities which must hold between the coefficients of the differential equation in any interval for the relative maxima of $|x|$ to form an increasing or decreasing sequence in that interval. (Author's summary.)

J. K. Hale (Saint Paul, Minn.).

Tatarkiewicz, Krzysztof. Contribution à la théorie des équations différentielles. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 29–36 (1957). (Polish and Russian summaries)

There are two "formulas" for the solution of the system $dx/dt=Ax$, where $x=x(t)$ is a vector and A is a constant matrix. One is based on the reduction of A to its Jordan canonical form, the other is based on a search for solutions of simple exponential form. While the relationship between these methods is obvious in the case of simple elementary divisors, the author discusses the situation in the more complicated general case and obtains some explicit formulas (believed to be new) for both complex and real general solutions.

D. C. Lewis (Baltimore, Md.).

Takahashi, Ken-ichi. Ein System von linearen homogenen Differentialgleichungen, welche auf zwei Parametern abhängen. Tôhoku Math. J. (2) **8** (1956), 258–267.

The author considers the two-dimensional system $y'=\lambda^\sigma\mu A(x, \lambda, \mu)y$ under the following assumptions: (i) $A(x, \lambda, \mu)$ is continuous for $x\in[a, b]$, $\lambda, \mu\geqq R>0$; (ii) $A(x, \lambda, \mu)\sim\sum A_{rs}(x)\lambda^{-r}\mu^{-s}$ as $\lambda, \mu\to\infty$ with A_{rs} sufficiently smooth in $[a, b]$; (iii) $A_{00}(x)$ has a double eigenvalue; (iv) $\mu=O(\lambda^\sigma)$ for some $\sigma\in[1, 1+\varepsilon]$ with arbitrarily small $\varepsilon>0$. He uses suitable transformations which reduce the system to one depending upon a single parameter and applies results of Hukuhara [Mem. Fac. Engrg. Kyushu Univ. **8** (1937), 249–280] to obtain formal asymptotic expansions of the solutions for large λ, μ. Details are too lengthy to be stated here.

H. A. Antosiewicz (Providence, R.I.).

Levin, J. J. The asymptotic behavior of the stable initial manifolds of a system of nonlinear differential equations. Trans. Amer. Math. Soc. **85** (1957), 357–368.

A previous investigation by the author [Duke Math. J. **23** (1956), 609–620; MR **18**, 305] is developed further; it concerns the behaviour as $\varepsilon\to0$ of solutions of the system $x'=f(t, x, y, \varepsilon)$, $\varepsilon y'=g(t, x, y, \varepsilon)$ whose initial values are close to those of a known solution $x=p(t)$, $y=q(t)$ of the degenerate system $x'=f(t, x, y, 0)$, $0=g(t, x, y, 0)$, and the hypotheses in the previous paper are retained. It is now shown that the solution with $x(0)=p(0)+a$, $y(0)=b$, where b lies on a certain "stable initial manifold" $S(a, \varepsilon)$, can be well approximated by using the boundary layer equation $dY/d\tau=g(0, p(0)+a, Y, 0)$ with the initial condition $Y(0)=B$, where B lies on a manifold $S^*(a)$ and $\tau=t/\varepsilon$. It can be inferred from this that $S(a, \varepsilon)$ approaches $S^*(a)$ as $\varepsilon\to0$. *G. E. H. Reuter* (Manchester).

Reissig, Rolf. Über die Beschränktheit der Lösungen einer nichtlinearen Differentialgleichung. Math. Nachr. **15** (1956), 375–383.

All solutions of $x''+kF(x')+g(x)=kp(t)$ $(0<k\leq1, t\geq0)$ satisfy $\limsup_{t\to\infty}(|x(t)|+|x'(t)|)\leq B$, where B depends only on the functions F, g, p but not on the parameter k, provided that the following conditions are satisfied: (1) $p(t)$ continuous, $m\leq p(t)\leq M$, with $M>m$; (2) $F(y)$ continuous, $F(y)\geq(M-m)(1+\varepsilon)\theta$ for $y\geq Y$, $F(y)\leq-(M-m)\theta$ for $y\leq-Y$, where $0<\theta\leq1$, $\varepsilon>0$, $Y>0$; (3) $g(x)$ continuous, $g(x)\geq kM+(1-k)(M-m)\theta-f+\eta$ for $x\geq X$, $g(x)\leq km-(1-k)(M-m)(1-\theta)-F-\eta$, where $\eta>0$, $X>0$, F and f being sup and inf of $F(y)$ in $|y|\leq Y$. This sharpens (considerably) a result by the reviewer [J. London Math. Soc. **27** (1952), 48–58; MR **13**, 844]. *G. E. H. Reuter.*

Littlewood, J. E. On non-linear differential equations of the second order. III. The equation $\ddot{y}-k(1-y^2)\dot{y}+y=b\mu k\cos(\mu t+\alpha)$ for large k, and its generalizations. Acta Math. **97** (1957), 267–308.

This is an informal introduction to work of the author and M. L. Cartwright which was summarized briefly in an earlier paper by these authors [J. London Math. Soc. **20** (1945), 180–189; MR **8**, 68]. In this introduction complicated and tedious details are omitted and attention is confined to the Van der Pol equation for large values of k, but similar results are announced for more general equations.

The results are classified according to $b>2/3$ or (*) $1/100\leq b\leq2/3-1/100$. The case where $b>0$ is very small is not treated. If $b>2/3$ and k is larger than a certain constant, which is a function of b and μ, then the equation of the title has a stable periodic motion of order 1 and period μ to which every trajectory converges. When (*) holds, the behavior of the trajectories is very complicated. For example, there may be both stable and unstable subharmonics, and while many of the trajectories converge to some stable subharmonic there may be, at the same time, uncountably many non-periodic limiting trajectories. Even finer details are produced but they would be difficult to describe in a review. *W. R. Utz.*

Pliss, V. A. An investigation of a non-linear system of three differential equations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 184–187. (Russian)

In the system of the third order

$$\dot{x}=y-f(x),\ \dot{y}=z-x,\ \dot{z}=-ax-bf(x) \tag{1}$$

let the non-linear part $f(x)$ satisfy a Lipschitz condition, and in addition let

$$f(0)=0,\ \frac{f(x)}{x}>a+b\frac{f(x)}{x}>0 \text{ for } x\neq 0. \tag{2}$$

The author examines various circumstances in relation to the values of the constants a, b. For some of these he uses a special Liapunov function of type: (quadratic form in $x, y, z)+\int_0^x f(\xi)d\xi$. For certain locations of a, b he finds sufficient, or else necessary and sufficient, conditions for asymptotic stability in the large; likewise for the existence of periodic solutions. The most interesting result is: Necessary and sufficient conditions for asymptotic stability of the origin in the large, whatever $f(x)$ satisfying (2), are one of the following: (1) $a<0$, $b>0$; (2) $a=0$, $0<b<1$; (3) $a>0$, $b<0$, $a^2+b(1-b)^2\leqq 0$.

S. Lefschetz (Mexico, D. F.).

Iakubovich, V. A. On a class of non-linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 44–46. (Russian)

This paper brings noteworthy complements to the theory of regulating systems and their stability developed by Lur'ie [Some non-linear problems in the theory of automatic regulation, Gostehizdat, Moscow-Leningrad, 1951; MR **15**, 707; see also Letov, Stability of non-linear regulating systems, Gostehizdat, Moscow, 1955; MR **17**, 487]. The author couches the results of Lur'ie in vector-matrix form. He also shows that a certain complement to Lur'ie's results leads to a sufficient stability condition which includes all those derived from a certain Liapunov function.

The basic Lur'ie equation is

$$\dot{x}=Ax+a\cdot\varphi(\sigma),\ \dot{\sigma}=(b, x)-\rho\varphi(\sigma),$$

where x, a, b are n-vectors, A is an $n\times n$ matrix and the other quantitities are scalars. In particular, $\rho>0$ and $\varphi(0)=0$, $\sigma\varphi(\sigma)>0$ for $\sigma\neq 0$. Finally, the characteristic roots of A are supposed to have negative real parts. The question is to find under what conditions the origin ($x=0$, $\sigma=0$) is stable in the large (stable, and every solution $\to 0$ as $t\to+\infty$). The Liapunov function utilized in the problem is always of the form

$$V=(Hx, x)+\int_0^\sigma \varphi(\sigma)d\sigma,\ H=H' \ (H \text{ is symmetric}).$$

In the general case a sufficient condition for stability in the large is provided. When $A=-\alpha I$ (I the unit matrix of order n, $\alpha>0$) there is given a necessary and sufficient condition for such stability. When $n=2$, and assuming the vectors a, $A^{-1}a$ linearly independent, in special coordinates we have

$$A=\begin{pmatrix}-\alpha_0 & 1\\ -\alpha_1 & 0\end{pmatrix},\quad a=\begin{pmatrix}1\\0\end{pmatrix},\quad b=\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\qquad (\alpha_0, \alpha_1>0).$$

Sufficient conditions for stability in the large are then

$$\Gamma^2=\rho+\beta_1>0,\ \alpha_0(\rho\alpha_0-\beta_0)>\alpha_1(\rho^2-\Gamma)^2.$$

S. Lefschetz (Mexico, D. F.).

Ergen, W. K.; Lipkin, H. J.; and Nohel, J. A. Applications of Liapounov's second method in reactor dynamics. J. Math. Phys. **36** (1957), 36–48.

By use of Lyapunov's second method the authors obtain very simply sufficient conditions for simple and asymptotic stability of the trivial solution of the equations of a homogeneous and of a heterogeneous reactor. For example, the latter's behavior, in case of n media, is described by the system

$$d\log P/dt=-\sum_1^n \alpha_j T_j,$$

$$\varepsilon_i dT_i/dt=\eta_i(P-1)-\sum_1^n X_{ij}(T_i-T_j) \qquad (1\leqq i\leqq n),$$

where α_i, ε_i, η_i, X_{ij} are constants, $X_{ij}=X_{ji}$ and $\sum\eta_j=1$. (I) If $\alpha_i/\varepsilon_i=\beta>0$ ($1\leqq i\leqq n$), then $P=1$, $T_i=0$ ($1\leqq i\leqq n$) is stable. (II) If $R=(r_{ij})$ is such that $R'\mathrm{diag}(\varepsilon_i)R=I$, $R'(A_{ij})R=\mathrm{diag}(d_i)$, where $A_{ii}=\sum_{j\neq i}X_{ij}$, $A_{ij}=-X_{ij}$ ($i\neq j$) and d_i are constants not all zero, and if $\sum r_{ji}\eta_j>0$, $\sum r_{ji}\alpha_j>0$, $d_i\geqq 0$ ($1\leqq i\leqq n$), then $P=1$, $T_i=0$ ($1\leqq i\leqq n$) is asymptotically stable. *H. A. Antosiewicz.*

Sideriades, L. Systèmes non linéaires du deuxième ordre. J. Phys. Radium (8) **18** (1957), 304–311.

The author gives a geometric description of the solutions of two second order autonomous differential equations in two independent variables and illustrates his discussion on some oscillators of physical importance.

H. A. Antosiewicz (Providence, R.I.).

Bielecki, Adam. Extension de la méthode du rétracte de T. Ważewski aux équations au paratingent. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 37–61 (1957). (Polish and Russian summaries)

Bielecki, Adam. Certaines propriétés topologiques des solutions des équations au paratingent. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 63–79 (1957). (Polish and Russian summaries)

By paratingent $\mathrm{Pt}(Z, P)$ of a set $Z\subset E_{n+1}$ at a point $P\in Z$ we mean, as usual, the set $\mathrm{Pt}(Z, P)$ of all straight lines l issuing from a fixed point, say ($t=-1$, $x_1=\cdots=x_n=0$), having the following property: there are two sequences $[Q_i]$, $[R_i]$ of points of Z with $Q_i\neq R_i$, $Q_i, R_i\to P$, and the straight lines Q_iR_i approach a line l' parallel to l. Let $m(P)$ denote a closed bounded convex set of points of the x-space E_n, $x=(x_1, \cdots, x_n)$, and suppose that $m(P)$ is an upper semicontinuous function of P in an open set $D\subset E_{n+1}$. Correspondingly, $M(P)$ denotes the set of straight lines issuing from $(-1, 0, \cdots, 0)$ to any point of $m(P)$. If $I: x=\varphi(t)$ denotes a continuous arc in D, then the relation (*) $\mathrm{Pt}(I, P)\subset M(P)$, $P\in I$, is a paratingent equation, and all arcs I satisfying (*) are said to be solutions of (*). If $m(P)$ is a single point for every P, then at every point of I the direction of the tangent is determined and (*) reduces to a differential system. Equation (*) has many of the properties of the differential systems, but, in general, there are infinitely many solutions I through every point $P\in D$. The two subarcs right and left of P of a solution I are denoted by I^+ and I^-. The function $M(P)$ is said to define a field. [For references on the subject see Bielecki, same Ann. **2** (1947), 49–106; MR **11**, 360.] Extensive research on the subject has been done by S. K. Zaremba, A. Marchaud, G. Bouligand, F. Roger in 1930–35. — In the present paper, the author extends to equations (*) the recent topological theory of T. Ważewski concerning the asymptotic properties of the solutions of a differential

system [see T. Ważewski, Ann. Soc. Polon. Math. **20** (1947), 279–313; Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 143–148; MR **10**, 122; **16**, 1109; F. Albrecht, ibid. **2** (1954), 315–318; MR **16**, 248; A. Pliś, ibid. **2** (1954), 415–418; MR **16**, 700; A. Bielecki, ibid. **4** (1956), 493–495; MR **18**, 897]. Though a uniqueness theorem has been always required in all research on Ważewski's theory, most concepts can be given in the present more general situation in essentially the same form (quill, consequent, point of ingress and of egress, right and left shadow). For instance, if D is an open set and $M(P)$ is defined in D, if ω is an open subset of D and F the boundary of ω in D, then a point $P_0 \in F$ is a point of egress if there is an I^- from P_0 which is in ω; P_0 is a point of strong egress if in addition all I^+ from P_0 in a neighborhood of P_0 are in $D-\omega$. Finally, P_0 is a point of strict egress if all I^- from P_0 are in B, and all I^+ from P_0 are in $D-\omega$ (in a neighborhood of P_0). By means of a new analysis, the following main result among others is obtained, which is quite similar to one given by Ważewski. Under convenient hypotheses on D and $M(P)$ (too detailed to be given here), the set $\tilde{S}$ of points of strict egress is a retraction of $B+\tilde{S}$ in $B+\tilde{S}$ by a quasihomotopic deformation, where B is the left shadow of $\tilde{S}$. By a process of regularization of the field $M(P)$, another theorem is proved assuring, under hypotheses, that the sets $B\times S$ and $S\times[0, 1]$ are homeomorphic, where S is the set of points of egress. As application of the theory, two theorems of H. Kneser [S.-B. Preuss. Akad. Wiss. **1923**, 171–174] and of J. Szarski [Ann. Soc. Polon. Math. **20** (1947), 161–168; MR **10**, 121] are derived. *L. Cesari* (Lafayette, Ind.).

Krasnosel'skiĭ, M. A.; Kreĭn, S. G.; and Sobolevskiĭ, P. E. On differential equations with unbounded operators in Banach spaces. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 19–22. (Russian)

The authors consider the equation $dx/dt=A(t)x+f(t, x)$, where the unknown function $x(t)$ is required to have values in a Banach space E. $A(t)$ and $f(t, x)$ are given operators in E, A being assumed closed and linear for each t, and $x(0)=x_0$, where x_0 belongs to the domain $D(A)$ of $A(0)$. The existence of a solution is established under the assumptions i) $A(t)dA^{-1}(t)/dt$ is bounded and strongly continuous with respect to t, ii) the resolvent satisfies $\|(\beta I-A(t))^{-1}\|\leq 1/(1+\beta)$, $\beta>-1$, and iii) the Fréchet derivatives $f_t'(t, x)$ and $f_x'(t, x)$ are continuous and satisfy a Lipschitz condition in x, f_t' with respect to the norm in E and f_x' with respect to the norm in the space of linear operators over E. Also under less restrictive assumptions than iii) the authors prove the existence of suitably defined generalized solutions. Their results constitute an extension and improvement of results obtained by Phillips [Trans. Amer. Math. Soc. **74** (1953), 199–221; MR **14**, 882] and Kato [J. Math. Soc. Japan **5** (1953), 208–234; MR **15**, 437] and permit application to a wide class of partial differential equations. *A. N. Milgram.*

Aronszajn, N. On a problem of Weyl in the theory of singular Sturm-Liouville equations. Amer. J. Math. **79** (1957), 597–610.

Let $p(t)>0$, $p(t)$ and $q(t)$ be continuous on $0\leq t<\infty$, and suppose that the differential equation $-(px')'+(q-\zeta)x=0$ is of the limit-point type. Then the equation and boundary condition $x(0)\sin\alpha-x'(0)p(0)\cos\alpha=0$ at $t=0$ determine a boundary value problem with a spectrum S_α for every α on $0\leq\alpha<\pi$. Weyl showed that the set of cluster points of S_α is independent of α and raised the question as to whether the continuous spectrum is also independent of α. This issue and related ones can be regarded as one-dimensional perturbation problems in a more general theory of finite-dimensional perturbations of self-adjoint operators (in which connection is given a reference to a paper by the author and W. F. Donoghue). The present treatment, however, is based on the original Weyl theory framework and on some recent results of Gel'fand and Levitan [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 309–360; MR **13**, 558]. The Weyl problem is settled as follows: If $p(t)\equiv 1$ and if the differential equation is of the limit-point type, then the continuous spectrum is not invariant under a change of boundary condition, although the absolutely continuous part is. An example of a boundary value problem with a pure point spectrum dense on $(-\infty, \infty)$ for one boundary condition and with purely continuous spectra for all other boundary conditions is given. Another example is constructed in which for one boundary condition there is a pure continuous singular spectrum while for any other boundary condition there is a pure point spectrum. *C. R. Putnam.*

★ **Rellich, Franz. Halbbeschränkte Differentialoperatoren höherer Ordnung.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 243–250. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Consider a symmetric ordinary differential operator acting on functions defined in a specified interval (a, b) and satisfying prescribed boundary conditions at the end points. First proving that the semi-boundedness of such an operator depends on the coefficients but not on the boundary conditions, the author next announces a sufficient condition for semi-boundedness at a or at b separately; this condition is a generalization of one due to P. Hartman [Duke Math. J. **15** (1948), 697–709; MR **10**, 376] in the second-order case. Then a necessary and sufficient condition is stated pertaining to an end point at which the coefficients admit a certain kind of power series expansion. As an application, the largest value of c is adduced such that the inequality $c\int r^{-4}|u|^2 d\tau\leq\int|\Delta u|^2 d\tau$ holds for all functions $u(x_1, \cdots, x_k)$, which are infinitely often differentiable and vanish in a neighborhood of the origin and a neighborhood of infinity; here the domain of integration is the entire $(x_1, \cdots, x_k)$-space, and r denotes Euclidean distance from the origin, and $d\tau$ the element of volume, in this space. Other results are also described and the history of the subject briefly summarized.

A. Douglis (College Park, Md.).

Reid, William T. Adjoint linear differential operators. Trans. Amer. Math. Soc. **85** (1957), 446–461.

Linear differential expressions of the form $L(y)=\sum_{\mu=0}^{n} p_\mu(x)y^{(\mu)}(x)$ are considered, where the functions $p_\mu(x)$ are in the space $\mathfrak{L}(a, b)$. If the domains of definition of $L(y)$ are chosen from various subspaces of the space $C^{(n)}(a, b)$, $L(y)$ represents a linear operator from the particular space considered into $\mathfrak{L}(a, b)$. The author considers particularly the space $C_0^{(n)}\subset C^{(n)}$ of functions for which $y^{(\alpha)}(a)=y^{(\alpha)}(b)=0$ $(\alpha=1, 2, \cdots, n-1)$ and the corresponding operator T_0 is defined by $L(y)$. If $D^*\subset\mathfrak{L}(a, b)$ is the set of functions $z(x)$ for which $\bar{z}(x)p_\mu(x)\in\mathfrak{L}$ $(\mu=1, 2, \cdots, n)$ and for which there exists a function $f_z\in\mathfrak{L}$ such that $(L(y), z)=(y, f_z)$ for all $y\in C_0^{(n)}$, then $T_0^*z=f_z$ with domain D^* is the adjoint of T_0. The author proves various

theorems about the structure of T_0 and T_0^* in case D^* is sufficiently large. In particular, if the polynomials $\{x^\lambda/\lambda!\}$ are in D^* for all non-negative integers λ, then T_0, T_0^* must have a certain canonical form. If, for some $\varepsilon>0$, $|p_\mu(x)|\geqq\varepsilon$ a.e. on (a, b), and $\{x^\lambda/\lambda!\}\subset D^*$, then D^* is the set of functions having continuous derivatives of the first $n-1$ orders with the $(n-1)$st derivative being absolutely continuous. Other results of this type are established and applications to Sturm-Liouville operators are given in the last sections of the paper.

R. E. Fullerton (College Park, Md.).

Krumhaar, Hans. Ein Satz über die Separation von Spektren bei gewöhnlichen selbstadjungierten Differentialoperatoren gerader Ordnung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1956**, 267–274.

The eigenvalue problem for a regular self-adjoint ordinary differential operator

$$L(u)\equiv\sum_{\nu=0}^{m}(p_\nu(x)u^{(m-\nu)}(x))^{(m-\nu)}=\lambda k(x)u(x),$$

with $p_0(x)\neq 0$, $k(x)>0$ on a finite or infinite interval (a, b), is considered. Let S denote its spectrum with respect to certain self-adjoint boundary conditions at a and at b. The same eigenvalue problem is also considered on intervals (a, c) and (c, b), some new self-adjoint boundary conditions being imposed at the interior point c. Let S_1 and S_2 denote the spectra of these two problems. The author proves the following separation theorem, which generalizes a result of K. G. Wolfson [Amer. J. Math. **72** (1950), 713–719; MR **10**, 946]. If the interval $\lambda'\leqq\lambda\leqq\lambda''$ contains at least $m+r$ points of one of the two sets S and S_1+S_2, it contains at least r points of the other. In the case of a finite interval (a, b) this is a special case of a result of the reviewer [Duke Math. J. **22** (1955), 1–14, p. 10; MR **16**, 824].

H. F. Weinberger.

Jacubovich, V. A. On the dependence of the eigenvalues of the boundary problem for the system of two differential equations on boundary conditions. Vestnik Leningrad. Univ. **12** (1957), no. 1, 201–206, 213. (Russian. English summary)

Let the 2-vector $x(t)$ satisfy $dx/dt=IH(t, \lambda)x$, where $I=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, and H is real and symmetric. The author is mainly concerned with the boundary problem $x(\tau)=\pm Cx(0)$, where C is real, $\det C=1$, $\mathrm{sp}\, C>0$. Conditions on H, including that $\partial H/\partial\lambda>0$, are quoted which ensure the existence of a spectrum satisfying

$$\cdots<\lambda_{-1}^-\leqq\lambda_{-1}^+<\lambda_0^-\leqq\lambda_0^+<\lambda_1^-\leqq\lambda_1^+<\cdots,$$

where the suffixes are related to the angle through which the eigen-vector rotates as t goes from 0 to τ. Inequalities connect the spectrum with that for $x(\tau)=C'x(0)$, $\det C'=1$, $\mathrm{sp}\, C'>0$, where $\Delta=\det(C-C')\geqq 0$, separate results being given for $\Delta>0$, $\Delta=0$. Only the barest indications of the proofs are given, which use a toral representation of the group of unimodular matrices [cf. V. A. Yakubovič, Mat. Sb. N.S. **37(79)** (1955), 21–68; MR **17**, 483; and I. M. Gel'fand and V. B. Lidskiĭ, Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 3–40; MR **17**, 482].

F. V. Atkinson.

See also: Difference Equations, Functional Equations: Redheffer. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Slobodyànskiĭ. **General Topology:** Schwartzman. **Mechanics of Particles and Systems:** Bradistilov. **Control Systems:** Klotter.

Partial Differential Equations

Saltykow, N. Contribution à l'étude du domaine d'existence des intégrales des équations aux dérivées partielles d'ordre supérieur. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N. S.) **9** (1956), 27–37. (Serbo-Croatian. French summary)

Il s'agit dans les lignes qui vont suivre d'étendre le domaine d'existence des intégrales étudiées. On le considère ordinairement comme assez petit environnant les valeurs initiales des variables [v. G. Valiron, Equations fonctionelles, Masson, Paris, 1945, p. 552; MR **7**, 297] et tel que parfois les intégrales deviennent inutilisables en pratique [v. J. Hadamard, Le problème de Cauchy, Hermann, Paris, 1932, p. 20]. Or, nous considérons „presque" tout le domaine d'analyticité des équations, sauf certains points, lignes et surfaces singuliers, s'introduisant par l'intermédiaire des éléments de Cauchy. Considérons, par exemple, l'équation partielle et son intégrale $p_{13}-y^2p_{23}=0$, $U=yz/(1-xy)$, les indices 1, 2, 3 correspondant aux dérivées partielles de la fonction U, prises respectivement par rapport à x, y et z.

L'équation considérée admet l'intégrale citée dans "presque" tout le domaine de régularité de l'équation, exceptée la surface du cylindre hyperbolique $xy=1$, de l'hyperespace à quatre dimensions. On démontre dans le présent travail que les intégrales étudiées des équations linéaires aux dérivées partielles du second ordre, à un nombre quelconque de variables indépendantes, ainsi que celles des équations non linéaires existent dans "presque" tout le domaine de régularité considéré, résultant des propriétés des équations étudiées.

Résumé de l'auteur.

Lewy, Hans. An example of a smooth linear partial differential equation without solution. Ann. of Math. (2) **66** (1957), 155–158.

Experience with linear partial differential equations has shown that they generally possess smooth local solutions provided the equations are sufficiently smooth. This paper produces the first example of a system with coefficients in C^∞ having no smooth solutions in any domain. Consider

$$(\varepsilon)\qquad Lu=F_\varepsilon(x_1, x_2, y),$$

where $u(x_1, x_2, y)=v+iw$, v, w real,

$$L=-\partial/\partial x_1-i\partial/\partial x_2+2i(x_1+ix_2)\partial/\partial y,$$

$$F_\varepsilon=\sum_{j=0}^{\infty}\varepsilon_j c_j^{-c_1}\psi'(y+2q_jx_1-2p_jx_2),$$

and: 1) $\psi(y)\in C^\infty$ is real, periodic and nowhere analytic, 2) $P_j=(p_j, q_j, y_j)$, $\{P_j\}$ dense in R^3, 3) $N_j=\{P: d(P, P_j)<\rho_j\}$, $\lim\rho_j=0$, 4) $c_j=\max[j, |p_j|, |q_j|]$, and 5) $\varepsilon=(\cdots\varepsilon_j\cdots)\in E$, the Banach space of bounded real sequences. One shows that if $E_{jnm}\subset E$ is the set on which (ε) has a solution on N_j with Hölder $(1/n, m)$ continuous derivatives, then E_{jnm} is non-dense. Hence $\bigcup E_{jnm}\neq E$.

The proof hinges on a lemma showing that if (ε) with $\varepsilon=(\cdots\delta_{jk}\cdots)$ has a solution $\in C^1$ on N_k, then $\psi(y)$ is analytic at $y=y_k$.

I. I. Kolodner.

Segre, Beniamino. Sui sistemi di equazioni differenziali lineari a coefficienti costanti. II, III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 395–403, 531–539.

In queste due Note l'A. prosegue le interessanti ricerche già iniziate in stessi Rend. (8) **20** (1956), 271–277

[MR 19, 33], occupandosi adesso dei sistemi di equazioni alle derivate parziali lineari a coefficienti costanti.

Egli studia nella Nota II i sistemi in una sola funzione incognita e perviene ad assegnare condizioni sotto le quali riescono sufficienti, per l'esistenza di una soluzione, le condizioni necessarie che si ottengono come conseguenze differenziali del sistema. Le anzidette condizioni si esprimono in eleganti termini geometrici relativamente alle ipersuperficie determinate dai poligoni caratteristici delle equazioni del sistema. Viene anche computato il grado di indeterminazione della soluzione generale del sistema e indicate condizioni sotto le quali detta soluzione si può calcolare con quadrature.

I risultati ottenuti nella Nota II vengono sfruttati nella Nota III per lo studio dei sistemi in più funzioni incognite. Fra i diversi notevoli teoremi dimostrati occorre ricordarne uno, assai generale, relativo ai sistemi di n equazioni in m incognite ($n<m$), il quale afferma la possibilità di ottenere, in opportune ipotesi per il sistema, le funzioni di ogni m-pla di soluzioni come combinazioni lineari dei risultati di certi operatori differenziali lineari a coefficienti costanti agenti su $m-n$ funzioni arbitrarie. Come applicazione dei teoremi stabiliti, l'A. risolve il problema della esistenza di una funzione potenziale secondo Sobrero [ibid. (6) **21** (1935), 448–454] per un sistema lineare a coefficienti costanti. Egli assegna delle assai ampie condizioni sufficienti per l'esistenza di questa funzione potenziale. Un esempio mostra che, non verificandosi le condizioni (sufficienti) dell'A., la funzione potenziale può non esistere [Fichera, ibid. (8) **2** (1947), 527–532; MR **9**, 239]. Vengono anche risoluti, per i sistemi alle derivate parziali, problemi analoghi a quelli considerati nella Nota I per i sistemi ordinari, relativi alla compatibilità di una sistema non omogeneo ed alla riduzione di questo ad un sistema di cui ogni equazione contiene una sola incognita. *G. Fichera* (Roma).

Trèves, François. Solution élémentaire d'équations aux dérivées partielles dépendant d'un paramètre. C. R. Acad. Sci. Paris **242** (1956), 1250–1252.

The author proves the following theorem. Let Λ be a topological space; let $P(\xi, \lambda)$ be a polynomial in ξ on R^n with coefficients which are continuous in λ and such that the degree of $P(\xi, \lambda)$ is independent of λ. Then we can find a continuous map $\lambda \to E(\lambda)$ of Λ into D' (space of distributions on R^n) so that for each λ we have $P(D, \lambda)E(\lambda)=\delta$, where $P(D, \lambda)$ is the partial differential operator corresponding to $P(\xi, \lambda)$ and δ is Dirac's measure.

In case Λ is a point, the result was proven by Malgrange [same C. R. **237** (1953), 1620–1622; MR **15**, 626] and the reviewer [Amer. J. Math. **76** (1954), 883–903; MR **16**, 834]. Actually Shirota has since proved a more general result [in the paper reviewed below], namely, that with Λ, P as above, if $\lambda \to S(\lambda)$ is a continuous map of Λ into D', then we can find a continuous map $\lambda \to T(\lambda)$ of Λ into D' so that $P(D, \lambda)T(\lambda)=S(\lambda)$ for all λ. The proof given here is, in principle, similar to that given by Malgrange and the reviewer, except that the author replaces the usual use of the Hahn-Banach theorem by an explicit construction. *L. Ehrenpreis* (Waltham, Mass.).

Shirota, Taira. On solutions of a partial differential equation with a parameter. Proc. Japan Acad. **32** (1956), 401–405.

Let $P(\partial/\partial x, \lambda)$ be a family of partial differential operators with constant coefficients on R^n parametrized by λ running through a separable locally compact space Λ with the degree of P_λ independent of λ. Suppose that T_λ is a continuous mapping of Λ into the space $\mathscr{D}'$ of distributions on R^n, and that for $\lambda \in \Lambda_0$, a closed subspace of Λ, there is given a continuous mapping S_λ of Λ_0 into $\mathscr{D}'$ satisfying the equation $P_\lambda S_\lambda = T_\lambda$. Then if the coefficients of P_λ depend continuously on λ, there exists a continuous mapping S' of Λ into $\mathscr{D}'$ such that $S_\lambda' = S_\lambda$ on Λ_0 while $P_\lambda S_\lambda' = T_\lambda$ for all λ. Similar results are stated for $\mathscr{E}'$, the space of distributions with compact support. The result is a direct extension of theorems concerning a single operator of Malgrange [C. R. Acad. Sci. Paris **237** (1953), 1620–1622; **238** (1954), 196–198; MR **15**, 626], Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853], and Ehrenpreis [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 756–758; MR **17**, 877]. For the parametrized case, it is related to the construction by F. Trèves of a fundamental solution continuously varying with the parameter [see the paper reviewed above]. *F. Browder.*

Mihaĭlov, L. G. A boundary problem of the type of Riemann for systems of first order differential equations of elliptic type. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 13–15. (Russian)

The author considers the elliptic system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = a(x, y)u + b(x, y)v,$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = c(x, y)u + d(x, y)v;$$

which may be written, in matrix form, as (*) $\partial U/\partial \bar{z} = A(z)U$ ($U=u+iv$); and for which a theory of functions, analogous to the ordinary function theory for the Cauchy-Riemann equations, has been constructed by I. N. Vekua [Mat. Sb. N.S. **31(73)** (1952), 217–314; MR **15**, 230] and L. Bers [e.g., Theory of pseudo-analytic functions, Inst. Math. Mech., New York, Univ., 1953; MR **15**, 211]. Using a formula of Vekua [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 181–184; MR **16**, 1114] which establishes a one to one correspondence between analytic functions $\varphi(z)$ and solutions $U(z)$ of the system (*), the author formulates an analogue for (*) of the generalized Liouville theorem concerning entire analytic functions of finite order at infinity. Using a method employed by F. D. Gahov [Izv. Fiz.-Mat. Obšč. Naučno-Issled. Inst. Mat. Meh. Kazan. Univ. (3) **14** (1949)] for the solution of the "ordinary" Riemann problem for analytic functions, the author solves the Riemann problem for (*), which consists in the determination of a piecewise-regular solution of (*), of finite order at infinity, and such that, for t on a finite number of smooth closed Jordan curves L, one has $U^+(t)=G(t)U^-(t)+g(t)$, where $G(t)\neq 0$ and $g(t)$ are given Hölder continuous functions on L, and the superscripts + and − have the same meaning as in the usual Riemann problem. More general boundary value problems for (*) are also treated. *J. B. Diaz* (College Park, Md.).

Danilyuk, I. I. Quasiharmonic and quasianalytic functions on surfaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 95–101. (Russian)

Complex-valued functions whose real and imaginary parts are solutions of elliptic systems in two independent variables have certain well-known properties related to the properties of analytic functions. It is natural to expect that this analogy will not be only of local character, so that the question arises of studying such systems "in the large", that is, on certain surfaces. In the present

article we give some results related to this question. See also Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 11–13 [MR **17**, 741]. *From the introduction.*

Danilyuk, I. I. On automorphic quasi-analytic functions on surfaces. Mat. Sb. N.S. **41(83)** (1957), 97–104. (Russian)

The author outlines certain results in the theory of elliptic systems of first order partial differential equations (*) $\partial u^1/\partial x^i=\alpha_i{}^j\ \partial u^2/\partial x^j$ $(i, j=1, 2)$, on an orientable surface R (or a part of it), which is thrice Hölder continuously differentiable, and where $\alpha_i{}^j$ is a tensor. It is known [e.g., Danilyuk, reference cited in review above] that the solutions $f=u^1+iu^2$ of (*) have various pseudo-analytic properties, and in the present paper the author discusses the automorphic character of certain solutions of (*) on R. *A. J. Lohwater* (Ann Arbor, Mich.).

Šabat, B. V. On an analogue of Riemann's theorem for linear hyperbolic systems of differential equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 5(71), 101–105. (Russian)

The problem, important for applications, is considered of the mapping of plane regions for hyperbolic systems of first order (analogous to the problem of conformal mapping for the elliptic system of Cauchy-Riemann). In the case of constant coefficients the system under consideration can be reduced to the form: $u_x=v_y$; $u_y=v_x$. The mapping problem can be solved only if the regions considered on each of the planes are of the same type in their situation with respect to the characteristics $x\pm y=c$ and their images $u\pm v=c$. It is proved that if the region D on the (x, y) plane is a region of the "half-plane type" or of the "strip type", then there exists a solution of the system which maps it onto a half-plane or, respectively, onto a strip of the (u, v) plane. The mapping depends on an arbitrary function. The results can be extended to systems with variable coefficients.

Z. Ya. Šapiro (RŽMat **1957**, no. 6321).

Pettineo, Benedetto. Sul prolungamento analitico delle soluzioni di talune equazioni a derivate parziali della fisica-matematica. Ann. Mat. Pura Appl. (4) **41** (1956), 221–255.

L'Autore dà una dimostrazione della possibilità di prolungare analiticamente, nello spazio S_3, una funzione armonica $u(P)$ attraverso una superficie analitica, sulla quale la $u(P)$ abbia una traccia analitica; estende pertanto dal piano allo spazio il classico teorema di Schwarz [una prima dimostrazione trovasi in Giraud, Ann. Sci. Ecole Norm. Sup. (3) **43** (1926), 1–128, p. 97]. Qui l'A. perviene alla tesi dimostrando che la $u(P)$ si può rappresentare con un potenziale di doppio strato avente momento analitico; la prolungabilità segue allora da un teorema di Schmidt.

Il procedimento seguito dall'A. era già stato introdotto, per il caso piano, dal recensore [Portugal. Math. **2** (1941), 173–176; MR **3**, 125], il quale non è a conoscenza di una precedente osservazione del prof. Picone: quanto afferma il Pettineo nella nota (3) è da ritenersi erroneo.

L. Amerio (Milano).

Landis, E. M. On the Phragmén-Lindelöf principle for elliptic equations. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 508–511. (Russian)

Vinogradov, V. S. On Neuman's problem for a partial differential equation of elliptic type. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 13–16. (Russian)

An application of the method of I. N. Vekua [same Dokl. (N.S.) **101** (1955), 593–596; MR **16**, 1115].

Sobolev, S. L. Sur quelques travaux soviétiques concernant les applications de l'analyse fonctionnelle aux équations différentielles. Czechoslovak Math. J. **6(81)** (1956), 289–310. (Russian. French summary)

Expository review without proofs of some applications of functional analysis (Hilbert space theory and the imbedding theorems of Sobolev and others) to boundary value problems, in particular to mixed problems.

L. Gårding (Lund).

Smirnov, M. M. Functional-invariant solutions of fourth-order equations with two independent variables. Vestnik Leningrad. Univ. **11** (1956), no. 7, 122–125. (Russian)

N. P. Erugin [Leningrad. Gos. Univ. Uč. Zap. **111**, Ser. Mat. Nauk **16** (1949), 142–166] gave necessary and sufficient conditions for the existence of functionally invariant solutions (i.e., functions $u(x, y)$ such that $F(u)$ is a solution, where F is an "arbitrary" function) of linear second order partial differential equations in two independent variables. Following Erugin's method, the author gives similar conditions for the fourth order equation

$$\sum_{i+j=1}^{4} a_{ij}\frac{\partial^{i+j}u}{\partial x^i\partial y^j}=0,$$

whose coefficients are thrice continuously differentiable functions of x and y. *J. B. Diaz.*

Galonen, L. M. On functionally invariant solutions of the wave-equation in an n-dimensional region. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 53–72. (Russian)

A functionally invariant solution of a partial differential equation is a solution such that an arbitrary function of it is again a solution. The determination of the solutions mentioned in the title consists in the solution of the system:

$$\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2=\left(\frac{\partial u}{\partial t}\right)^2,\quad \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i{}^2}=\frac{\partial^2 u}{\partial t^2}.$$

In a previous paper the author considered this question for $n=2$, and $n=3$ [Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), 179–182; MR **18**, 487], simplifying an earlier procedure of N. P. Erugin [Leningrad Gos. Univ. Uč. Zap. **96**, Ser. Mat. Nauk **15** (1948), 101–134]. The present paper shows the direct applicability of the author's method to arbitrary n. *J. B. Diaz* (College Park, Md.).

Lavrent'ev, M. M. On the problem of Cauchy for linear elliptic equations of the second order. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 195–197. (Russian)

The author has previously considered Cauchy's problem for Laplace's equation [same Dokl. (N.S.) **106** (1956), 389–390; MR **18**, 799], and E. M. Landis [ibid. **107** (1956), 640–643; MR **17**, 1212] has considered Cauchy's problem for an arbitrary linear elliptic equation with sufficiently smooth coefficients. In this note the author generalizes to an arbitrary elliptic equation his earlier results for Laplace's equation. He obtains, as a consequence, the following theorem, which extends Landis' results. Sup-

pose that $u(x)$ is a solution of the linear elliptic equation

$$(*) \qquad \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}+cu=0,$$

in the unit sphere G, where a_{ij}, $\partial a_{ij}/\partial x_k$, $\partial^2 a_{ij}/\partial x_m \partial x_k$, b_i, $\partial b_i/\partial x_k$, c, are continuous in G and bounded in absolute value by unity. Further, there is a constant α such that $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqq \alpha > 1$, for $\sum_{i=1}^{n}\xi_i^2=1$. Let S be the boundary of G, and let γ_l be the subset of S defined by the inequalities $\sum_{i=1}^{n} x_i^2=1$, $\sum_{i=2}^{n} x_i^2 \leqq l^2$, $x_1>0$. Then, there exist constants C_1 and C_2, depending on n and α alone, such that, for any $\varepsilon>0$, and $0<l<\frac{1}{2}$, any solution u of the elliptic equation (*), continuous with its first partial derivatives on G plus its boundary, which satisfies the inequalities $|u|_S \leqq 1$, $|u|_{\gamma_l} \leqq \varepsilon$, $|\partial u/\partial n|_{\gamma_l} \leqq \varepsilon$, must also satisfy

$$|u(0)| \leqq C_1 \varepsilon^{C_2 l^{2+\delta}} \quad (\delta>0).$$

J. B. Diaz (College Park, Md.).

Lavruk, B. R. Condition of solubility of a boundary value problem for a system of linear differential equations of second order of elliptic type. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 23–25. (Russian)

Let A (supposed elliptic) and B be differential operators, with certain differentiability conditions in a region D of points $x=(x_1, \cdots, x_n)$, of the form

$$A\left(x, \frac{\partial}{\partial x}\right)=\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} A_i(x)\frac{\partial}{\partial x_i}+A(x),$$

$$B\left(y, \frac{\partial}{\partial x}\right)=\sum_{i=1}^{n} B_i(y)\frac{\partial}{\partial x_i}+B(y),$$

where the coefficients are matrices of order p. Consider the boundary problem for $u(x)$:

$$A(x, \partial/\partial x)u(x)=0 \quad (x \in V), \quad \lim_{x \to y} B(y, \partial/\partial x)u(x)=f(y)$$

($y \in S$), where V is a convex region in D together with its smooth boundary S and $f(y)$ is a given continuous colum vector. (Some details of the formulation are here omitted.) Then necessary and sufficient conditions for the existence of a solution (and some properties of the solution) are given in terms of the solutions $\lambda(x)$ of an integral equation

$$\lambda(y)+\int_S \lambda(z)B(z, \partial/\partial z)G(z, y)d_z S=0$$

where $G(z, y)$ is a function of Green type.

Lavruk, B. R. On a certain boundary problem for a system of elliptic equations. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 214–219. (Ukrainian. Russian summary)

Results closely related to those of the article reviewed above.

Nitsche, Johannes. Untersuchungen über die linearen Randwertprobleme linearer und quasilinearer elliptischer Differentialgleichungssysteme. I, II. Math. Nachr. **14** (1955), 75–127, 157–182.

These papers are devoted largely to a study of boundary value problems of the form

$$(1) \qquad p(s)u+q(s)v=h(s), \quad p^2+q^2=1,$$

for functions $w=u(x, y)+iv(x, y)$ which are solutions of an equation of the form

$$(2) \qquad w_{\bar{z}}+A(z)w+B(z)\bar{w}=F(z, w)$$

in a simply connected, bounded domain T whose boundary S consists of a curve with a Hölder continuously turning tangent. (1) may also be written as

$$(1') \qquad R^{\sigma}(w)\equiv \mathrm{Re}\{e^{-i\sigma(s)}w(z(s))\}=h(s),$$

where $e^{-i\sigma(s)}=p-iq$. The case where (2) is linear, i.e., where the right-hand side depends only on z, has been considered by various authors (W. Haack, G. Hellwig, I. N. Vekua). Vekua, in fact, solved this problem for multiply connected domains [Mat. Sb. N.S. **31**(73) (1952), 217–314; MR **15**, 230].

The index of the boundary condition (1') is the integer

$$(3) \qquad n=\frac{1}{2\pi}\{\sigma(s_0+L)-\sigma(s_0)\},$$

where L is the length of the boundary curve.

The author first turns to a re-examination of the boundary value problem for linear equations. He proves that, for $n \geqq 0$, the modified boundary value problem

$$w_{\bar{z}}+Aw+B\bar{w}=F(z), \quad R^{\sigma}(w)=h(s),$$

$$(4) \qquad \int_S h_j(s)R^{\tau}(w)ds=k_j \quad (j=0, 1, \cdots, 2n),$$

where k_j are prescribed constants, $\tau(s) \not\equiv \sigma(s) \pmod{\pi}$ and $h_0(s)$, $h_1(s)$, $\cdots$, $h_{2n}(s)$ are Hölder continuous periodic functions of period L whose zeros satisfy certain simple conditions, has one and only one solution. For $n<0$, there are no solutions in general. However, the use of the adjoint problem

$$(5) \qquad \omega_{\bar{z}}-A\omega-\bar{B}\bar{\omega}=0, \quad R^{-(\sigma+\delta)}(\omega)=0 \qquad \left(e^{i\delta}=\frac{dz}{ds}\right),$$

enables one to prove that there exists exactly one linear combination

$$\sum_{j=0}^{2(|n|-1)} k_j h_j(s)$$

such that the problem (4), with the modified boundary condition

$$R^{\sigma}(w)=\tilde{h}(s)\equiv h(s)-\sum_{j=0}^{2(|n|-1)} k_j h_j(s)$$

has a unique solution.

Integral representations for these solutions are obtained with the aid of a pair of generalized Green's functions. It is not difficult to show that the Green's functions of problem (4) (with $n \geqq 0$) and those of problem (5) are connected by a simple relation. The existence of these Green's functions is first proved for a boundary value problem (4) with $A=B=0$. The differences between the corresponding Green's functions for the general problem (4) and the simple problem satisfy an integral equation of Fredholm type. Existence of a solution is readily established and the uniqueness follows easily.

With a view toward application of these results to nonlinear equations, the author establishes certain estimates for solutions $w(z)$ of (4), (5) in terms of F, h and the constants k_j. (The constants only appear for $n \geqq 0$.)

In the second paper the author uses the integral representation and the estimates, together with Schauder's fixed point theorem, to prove the existence of a solution for the modified boundary value problems for the more general equation (2) under suitable restrictions for the function $F(z, w)$. A similar method is used for boundary value problems involving the system

$$u_x=[-F(x, y; u, v)v_x+E(x, y; u, v)v_y](EG-F^2)^{-1},$$
$$u_y=[-G(x, y; u, v)v_x-F(x, y; u, v)v_y](EG-F^2)^{-1}.$$

The uniqueness theorems are proved with the aid of the uniqueness theorem of I.

By the method of successive approximations it is shown that if the boundary value problem for (2) has a solution w^0, then it also has a solution w in a neighborhood of w^0, when the quantities F, h, and k_j are perturbed. Under suitable conditions, similar perturbation and uniqueness theorems are also derived for a general quasilinear system. Application of the results to the equations $\psi=f(x, y; \psi_x, \psi_y)$ are given. *W. Koppelman.*

Finn, Robert. Growth properties of solutions of non-linear elliptic equations. Comm. Pure Appl. Math. 9 (1956), 415–423.

This is an exposition of some previously published results of the author concerning elliptic differential equations of the form

$$(1)\quad a(x, y, \varphi, p, q)\varphi_{xx}+2b(x, \cdots, q)\varphi_{xy}+c(x, \cdots, q)\varphi_{yy}=0 \qquad (ac-b^2\equiv 1),$$

$$(2)\quad [A(x, \cdots, q)]_x+[B(x, \cdots, q)]_y=0.$$

Here $\varphi(x, y)$ is the unknown function, $p=\varphi_x$, $q=\varphi_y$ are its first derivatives. The author is mainly interested in equations whose solutions exhibit properties similar to those of the minimal surface equation such as: (a) (S. N. Bernstein) a minimal surface $\varphi(x, y)$ defined on the whole (x, y)-plane is itself a plane; (b) (L. Bers) a single-valued solution admits no isolated singularities. The minimal surface equation can be expressed in both forms (1), (2), with $a:b:c=(1+q^2):-pq:(1+p^2)$, $A=p(1+p^2+q^2)^{-\frac{1}{2}}$, $B=q(1+p^2+q^2)^{-\frac{1}{2}}$.

Associated with solutions of equation (1) or (2) in a natural way are certain Riemannian metrics on the plane, such as R_0: $ds_0^2=dx^2+dy^2$; R_1: $ds_1^2=c\,dx^2-2b\,dx\,dy+a\,dy^2$; R_2: $ds_2^2\sim(1+p^2)dx^2-2pq\,dx\,dy+(1+q^2)dy^2$; R_4: $ds_4^2\sim(1+B^2)dx^2-2AB\,dx\,dy+(1+A^2)dy^2$, and the deviation from the minimal surface equation is expressed by saying that the appropriate metric, R_1 or R_4, is quasi-conformally related to R_2 — written $R_1\sim R_2$.

Typical theorems: I. The result of Bernstein holds for equations (1) satisfying $R_1\sim R_2$ and some additional conditions, which the author believes to be unnecessary. II. If φ is a solution of (1) in the unit circle, and S is the area of the surface $\varphi=\varphi(x, y)$, then $R_1\sim R_2$ implies that the slope of the surface at the origin is bounded by $C_1\exp(C_2S)$, where C_1 is an absolute constant, and C_2 depends on the maximal dilatation between the metrics R_1, R_2. III. Under certain hypotheses on (2), including $R_0\sim R_4$, $R_1\sim R_2$, one may obtain an explicit bound for $|\operatorname{grad}\varphi|$ in every compact subset provided that $|\varphi|$ is bounded; hence the Dirichlet problem can be solved for arbitrary continuous boundary values on a convex plane curve.

Note that no three-point condition is imposed on the boundary curve. An interesting example is given of an equation (2) satisfying all the conditions of III except $R_1\sim R_2$, and for which the Dirichlet problem cannot in general be solved for continuous boundary values on a convex curve.

Other results are mentioned, including extensions of (b) and some results for inhomogeneous equations.

L. Nirenberg (New York, N.Y.).

Schechter, Martin. On estimating elliptic partial differential operators in the L_2 norm. Amer. J. Math. 79 (1957), 431–443.

Let A be a linear mth order partial differential operator with continuous complex-valued coefficients defined on the closure of a domain $G\subset R^n$ whose boundary is of class C^m. Let $C^r(G)$ denote the class of infinitely differentiable complex-valued functions on $\bar{G}$ whose derivatives of order less than r vanish on the boundary. Necessary and sufficient conditions are given in order that an inequality of the form

$$(*)\quad \int_G |D^m u|^2dx\leq K\int_G (|Au|^2+|u|^2)dx \text{ for } u\in C^r(G)$$

hold, where D^m denotes the generic derivative of order m. It is shown that if $n>2$ or if A is real, then (*) holds if and only if A is elliptic on $\bar{G}$ and $r\geq\frac{1}{2}m$. When $n=2$ and A is not real, the condition is slightly more complicated to state.

Browder [Bull. Amer. Math. Soc. **62** (1956), 381], Guseva [Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1069–1072; MR **17**, 161], and Nirenberg [Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742] had earlier established that if the real part of A is elliptic and $r\geq\frac{1}{2}m$, then (*) holds. However, the present proof is different (at least from the proofs of Browder and Nirenberg — the reviewer has not seen the work of Guseva). It uses Fourier transforms and is based mainly on the ideas developed by Aronszajn for the proof of his well-known coerciveness inequality. The proof of the necessity of the conditions is obtained by extending a simple example of Aronszajn.

There is one rather annoying misprint in the paper (and some others which are not very annoying); it occurs in the definition of the function g_t. *K. T. Smith.*

Lopatinskiĭ, Ya. B. Dependence of the solutions of a system of linear second order differential equations of elliptic type on the coefficients of the system. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 211–213. (Ukrainian. Russian summary)

This article considers a differential operator

$$A\left(x, \frac{\partial}{\partial x}\right)=\sum_{k,l=1}^{n} A_{kl}(x)\frac{\partial^2}{\partial x_k\partial x_l}+\sum_{k=1}^{n} A_k(x)\frac{\partial}{\partial x}+A(x)$$

with sufficiently smooth matrix coefficients under the assumption of unique solvability (for this operator) of the Dirichlet problem for a certain convex region bounded by a surface of Lyapunov type. It is shown that for an operator of the same type whose coefficients, together with certain of their derivatives, are sufficiently (uniformly) close to the corresponding quantities of the above operator, the Dirichlet problem is solvable and its solution is uniformly close to the solution of the Dirichlet problem for the above operator. *Author's summary.*

Flekser, M. Š. On the spectral function of the operator

$$-\sum_{i,k=1}^{3}\frac{\partial}{\partial x_i}\left[a_{ik}(x_1, x_2, x_3)\frac{\partial u}{\partial x_k}\right]+c(x_1, x_2, x_3)u.$$

Mat. Sb. N.S. **40(82)** (1956), 3–22. (Russian)

Let

$$Bu=-\sum_{i,k=1}^{3}\frac{\partial}{\partial x_i}\left[a_{ik}(x_1, x_2, x_3)\frac{\partial u}{\partial x_k}\right]+c(x_1, x_2, x_3)u$$

be an elliptic operator, operating on functions u defined in three-dimensional Euclidean space E. Let A be a self-adjoint realization of B in the Hilbert space $L^2(E)$. Closely following A. Ya. Povzner [Mat. Sb. N.S. **32(74)** (1953), 109–156; MR **14**, 755] the author constructs the resolvent kernel $\mathfrak{H}(x, y, \lambda)$ (Im $\lambda\neq 0$) of A and applies this

construction to a study of the properties of the spectral function. The results are identical with the corresponding ones of Povzner [loc. cit.] in the case $a_{ik}(\cdot)=\delta_{ik}$.

L. Gårding (Lund).

Mohr, Ernst. Eigenwerte gekoppelter nicht-selbstadjungierter elliptischer Differentialgleichungen zweiter Ordnung. Math. Nachr. **16** (1957), 1–49.

The major portion of this paper is concerned with the boundary problem involving the pair of mutually adjoint elliptic partial differential equations

$$(*) \qquad L[u]+\lambda v=0, \quad M[v]+\lambda u=0,$$

where $L[u]=\Delta u+au_x+bu_y+cu$, $M[v]=\Delta v-(av)_x-(bv)_y+cv$, and the conditions $u=0$, $v=0$ on the boundary Γ of the compact region G of consideration. Utilizing the proper functions and proper values of an associated boundary problem involving a single self-adjoint elliptic differential equation, the author derives the results of Geppert [Math. Ann. **95** (1926), 519–543; **98** (1927), 264–272] on the asymptotic distribution of proper values of (*) under weaker differentiability conditions on the coefficient functions $a(x, y)$, $b(x, y)$, $c(x, y)$ than those imposed by Geppert. Correspondingly, if $L[u]$ is the general elliptic differential expression $\alpha u_{xx}+2\beta u_{xy}+\gamma u_{yy}+au_x+bu_y+cu$, and $M[v]$ is the adjoint of $L[u]$, the results of Krall [Amer. J. Math. **57** (1935), 907–917] are obtained under weakened differentiability conditions on the coefficients.

W. T. Reid (Los Angeles, Calif.).

Petrovskiĭ, I. G. Some remarks on my papers on the problem of Cauchy. Mat. Sb. N.S. **39(81)** (1956), 267–272. (Russian)

In the author's important paper in Mat. Sb. N.S. **2(44)** (1937), 815–868, there is an error on pages 823–824 which was called to his attention by J. Schauder [see also J. Leray, Hyperbolic differential equations, Inst. Advanced Study, Princeton, 1953; MR **16**, 139; **18**, 313]. The present note shows how by exercising care, but without essential change in method, the error can be corrected.

A. N. Milgram (Minneapolis, Minn.).

Lions, J. L. Opérateurs de Delsarte et problèmes mixtes. Bull. Soc. Math. France **84** (1956), 9–95.

Let Λ_x denote an elliptic differential operator of arbitrary order with variable coefficients defined on an open set of the n-dimensional euclidean space. The mixed problems for the hyperbolic operator

$$(1) \qquad \Lambda_x+D^2+r(t)D+s(t) \quad (D=\partial/\partial t)$$

aro reduced to those for the hyperbolic operator

$$(2) \qquad \Lambda_x+D^2.$$

The integral transformation X in the space of functions,

$$Xf(x)=f(x)+\int_a^x X(x, y)f(y)dy \quad (a\leqq x, a\leqq y\leqq x),$$

is called a Delsarte operator for $L=D^2-q(t)$ if $XL=D^2X$ [C. R. Acad. Sci. Paris **206** (1938), 1780–1782; Acta Math. **69** (1938), 259–317]. The notion is extended to the space of scalar- and vector-distributions of Schwartz. Thus, by the isomorphism X of a subspace of the corresponding space of distributions, we have $D^2=XLX^{-1}$. In this way, a method of calculus is developed to obtain solutions of the mixed problems for (1) from those of (2); the latter problems have been treated by the author in his thesis [Acta Math. **94** (1955), 13–153; MR **17**, 745].

Chapter II is devoted to the singular case

$$(1') \qquad \Lambda_x+D^2+\frac{2k+1}{t}D.$$

The corresponding Delsarte operator was discussed earlier by Delsarte himself for the case when k belongs to a properly chosen band in the complex plane. The author extends the result to the case of arbitrary complex k, by a method of analytical continuation analogous to that of the M. Riesz potential. Thus the Delsarte operator for $D^2+t^{-1}(2k+1)D$ is defined by a kernel distribution of Schwartz.

K. Yosida (Tokyo).

Lavrent'ev, M. M. On a boundary problem for a hyperbolic system. Mat. Sb. N.S. **38(80)** (1956), 451–464. (Russian)

The problem of plane irrotational hypersonic flow of gas from left to right in $0\leqq y\leqq y(x)$ is reduced by the author to the study of the non-linear functional equation

$$f(u)=\phi\left\{\int_0^u \cos f(u_1)\cos\left[f(u_1)-2\sum_{k=1}^{\infty} f(u_1-2k)+\delta\right]du_1+c\right\},$$

where f is the unknown function, ϕ is the angle of inclination of the tangent to the curve $y=y(x)$, δ is a constant depending on the velocity of the flow at infinity, and c is an unknown constant depending on f.

If (a) $|\phi|<\pi/2$, (b) $\lim_{x\to-\infty}\phi(x)(4+\varepsilon)^{-x}=0$, (c) $\phi(x)$ satisfies a Lipschitz condition on the whole infinite line, the author proves that the above equation has a unique solution. If the function f so determined does not satisfy either of the inequalities

$-\delta<\sum_{k=0}^{\infty}[f(u+v-2k-1)+f(u-v-2k-1)]<\pi/2-\delta$,

$-\pi/2-\delta<\sum_{k=0}^{\infty}[f(u+v-2k-1)+f(u-v-2k-1)]<-\delta$,

then the flow in question does not exist. Also, if f satisfies one or both of these conditions, then there exists one or two flows, respectively.

A. O. Bicadze (RŽMat **1957**, no. 5595).

Asral, Bediz. On the solution of the Cauchy problem for parabolic equations. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 65–83. (Turkish summary)

Expressions of the form

$$\sum_{\nu=1}^{N} P_\nu(x)\frac{\partial}{\partial x}v(x-a, t)=u(x, a, t),$$

where $v(x, t)$ is the fundamental solution of the heat equation, are manipulated in a formal way to obtain solutions $\varphi(x, t)=\int_{-\infty}^{\infty}u(x, a, t)\Phi(a)da$ of $\partial^2\varphi/\partial x^2-\partial\varphi/\partial t+k(x)\varphi=0$ for $k(x)$ related to the functions $P_\nu(x)$ and the initial function $F(x)=\sum P_\nu(x)\Phi^{(\nu)}(x)$.

A. N. Milgram (Minneapolis, Minn.).

Slobodeckiĭ, L. N. Generalized solutions of parabolic and elliptic systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 997–1000. (Russian)

This note is concerned with differentiability, and estimates on the derivatives, of weak or generalized solutions of parabolic and elliptic systems of equations. Let

$$Lu\equiv B(x, t, \partial/\partial x)u-\partial u/\partial t=f(x, t)$$

be defined in the strip $E_n\times[0, T]$, where u and f are N-vectors of complex valued functions and B is an elliptic operator of order $2p$ having complex coefficients depending on x and t. The results are based on the known estimates for the fundamental matrix solution G [Eĭdelman, same Dokl. (N.S.) **98** (1954), 913–915; MR **17**, 857] and require the coefficients of L to be differentiable enough to make these estimates valid.

Let Q be a cylindrical domain, $Q=\Omega\times[0, T]$, where $\Omega\subset E_n$ has smooth boundary. Setting

$$\|u\|_{L_q(Q)}=\Big(\int_Q |u(x, t)|^q dtdx\Big)^{1/q},$$

the author asserts that if u is a weak solution of $L(u)=0$ defined in Q and having finite $L_q(Q)$ norm, then u is continuously differentiable in the interior of Q for all orders listed in the operator L. Moreover, a sequence of weak solutions $u_{(m)}$ which are Cauchy convergent in the $L_q(Q)$ norm must be uniformly convergent in any closed bounded subdomain Q' in the interior of Q. The same holds for the derivatives of $u_{(m)}$ if the derivative is one occurring in L. The vector function $u_1(x, t)=\int_0^t ds\int_\Omega G(x, t, y, s)f(y, s)dy$ is called a parabolic volume potential with density $f(y, s)$. Postulating Dini-type conditions on f permits an explicit representation of $D_x^{2p}u_1(x, t)$ in terms of G and its derivatives. Under the weaker condition $\|f(x, t)\|_{N_q(Q)}=\int_0^t \|f\|_{L_q(\Omega)}dt<\infty$, and setting $w(x, t)=\int_0^t u_1(x, s)ds$, it is shown that generalized derivatives of w satisfy $\|D_x^r w(x, t)\|_{L_q(\Omega)}\leqq C\cdot\|f(x, t)\|_{N_q(\Omega)}$ for $0\leqq r\leqq 2p-1$. By assuming independence of t, the preceding yields a generalization of results of Calderón and Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637].

A. N. Milgram (Minneapolis, Minn.).

Slobodeckiĭ, L. N. Theory of the potential for parabolic equations. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 19–22. (Russian)

Let $u_t-Mu=f(x, t)$ be a parabolic equation defined in the strip $E_n\times[0, T]$, where M is an elliptic operator of second order having coefficients which are complex-valued functions of x and t. The first and second boundary problems for a domain Ω with smooth boundary are solved by using jump relations analogous to those of a potential due to a single and double layer distribution. The coefficients a_{ij} in M are assumed to have second derivatives with respect to the x, which satisfy a uniform Hölder condition in x. The a_{ij} are assumed also uniformly Hölder continuous in t. Similar assumptions are made on the coefficients of the lower order terms. The results are used to construct and estimate the Green's function. Applying these to the case where M is independent of t, it is shown that if u is a solution of $Mu+\lambda u=0$, with zero boundary values, then $u\equiv 0$ if $\operatorname{Re}\lambda$ is sufficiently small, and

$$|u(x, \lambda)|\leqq C(\operatorname{Re}\lambda)^{\frac{1}{2}n}\Big[\int_\Omega |u(y, \lambda)|^2 dy\Big]^{\frac{1}{2}},$$

$$|D_x^r u(x, \lambda)|\leqq C'(\operatorname{Re}\lambda)^{\frac{1}{2}n+\frac{1}{2}r}\Big[\int_{\Omega'} |u(y, \lambda)|^2 dy\Big]^{\frac{1}{2}},$$

where Ω' is an arbitrary subdomain in the interior of Ω and C' depends only on Ω'. *A. N. Milgram.*

Koubek, Ladislav. Über eine gewisse Eigenschaft der Lösungen einer parabolischen partiellen Differentialgleichung. Czechoslovak Math. J. **5(80)** (1955), 91–98. (Russian. German summary)

If $n+1$ solutions $x^i(u, v)$ of a parabolic equation

$$\frac{\partial^2 x}{\partial u^2}=a(u, v)\frac{\partial x}{\partial u}+b(u, v)\frac{\partial x}{\partial v}+c(u, v)x \quad (b\neq 0)$$

satisfy a quadratic relation $\{x, x\}=\sum_{i,k} a_{ik}x^i x^k=0$ with constant coefficients a_{ik}, then the solutions are linearly dependent. All the functions are assumed analytic. It is shown that if the functions $x^i(u, v)$ satisfy $\{x, x\}\equiv 0$, then the surface $x^i=x^i(u, v)$ is contained in an n-dimensional vector subspace of E_{n+1}. The author gives no indication of the connection, if any, in which this rather curious result arises. *A. N. Milgram* (Minneapolis, Minn.).

Èĭdel'man, S. D. Behavior of solutions of the heat equation in the neighborhood of an isolated singular point. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 207–210. (Russian)

Let $u(x, t)$ be a solution of the heat equation $\Delta u=\partial u/\partial t$ for $x\in V$, a domain in E_n, and $t_1<t<T$. Let u be regular in $V\times(t_1, T)$ except at the isolated point (x_0, t_0). The author shows that under certain reasonable conditions u must be a linear combination of derivatives of $U_0(t-t_0, x-x_0)$ (where U_0 is the fundamental solution of the heat equation) plus a solution regular in $V\times(t_1, T)$. Taking x_0 to be the origin, the class K_p is defined as the totality of functions continuous in $V-\{0\}$ and such that for $p>0$, $|\varphi(x)|\cdot|x|^p$ is bounded, for $p=0$, $|\varphi(x)|/|\ln x|$ is bounded and for $p<0$, $|\varphi(x)|$ is bounded. K_{p-0} is the union of K_q for $q<p$. Taking $t_0=0$, conditions such as

$$\int_{t_1}^T |t|^s|u(x, t)|dt\in K_{N+1-2s-0},\quad \int_{t_1}^T |t|^s|\partial u/\partial R|dt\in K_{N+2-2s-0}$$

for $s=0, 1, \cdots, r$ and $r=[(N-1)/2]+1$, imply u is a linear combination of spatial derivatives of $U_0(t, x)$ up to order $N-1$ plus a regular solution. These results lead to a characterization of removable singularities, and the remark that the fundamental solution is characterized by the requirement that it be a solution with a singularity of minimal order. The results are analogous to those describing the behavior of harmonic functions in the neighborhood of an isolated singularity. For very general elliptic systems similar results were obtained by Ya. B. Lopatinsky [Dokl. Akad. Nauk SSSR (N.S.) **79** (1951), 727–730; MR **16**, 827]. The author asserts that analogous results can be obtained for general p-parabolic (in the sense of Petrowsky) equations by using known properties of the fundamental matrix solution of such equations.

A. N. Milgram (Minneapolis, Minn.).

Aronson, D. G. Linear parabolic differential equations containing a small parameter. J. Rational Mech. Anal. **5** (1956), 1003–1014.

The author investigates the first boundary value problem for the parabolic equation

$$L_\varepsilon(u)\equiv\varepsilon u_{xx}+a(x, y)u_x-b(x, y)u_y+c(x, y)u=d(x, y)$$

in a suitable region R of the (x, y)-plane, where $\varepsilon\geqq 0$ is a parameter and $b(x, y)>0$. The results are analogous to those of Levinson [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439] and an extension of S. L. Kamenomostskaya [Mat. Sb. N.S. **31(73)** (1952), 703–708; MR **14**, 877]. Call a regular multilateral M the intersection of R and the strip between two characteristics of $L_0(u)$. It is shown that a solution $u(x, y, \varepsilon)$ in R when restricted to M has the form $u=v+z+w$, where v is a solution of $L_0(v)=d$, w is $O(\varepsilon)$ in M, and z is a boundary layer term similar to that obtained by Levinson. Except in the construction of z, the results are based on the maximum principle, which provides a simpler and more elegant method of treating the problem. *A. N. Milgram* (Minneapolis, Minn.).

Halilov, Z. I. On the investigation of asymptotic stability of solutions of boundary problems for partial differential equations. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 375–378. (Russian. Azerbaijani summary)

The stability of solutions, in the sense of Poincaré-Lyapunov, of nonlinear parabolic partial differential

equations has been investigated by Bellman [Trans. Amer. Math. Soc. **64** (1948), 21–44; MR **10**, 43], Prodi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10** (1951), 365–370; MR **13**, 351], and Narasimhan [J. Rational Mech. Anal. **3** (1954), 303–313; MR **15**, 799]. The author considers a generalized equation of this type in the form of a nonlinear operator equation $du/dt = A(t)u+F(t, u)$. He states a number of results concerning stability with respect to various norms, such as that of L^2 or B. Proofs of these results are omitted or briefly sketched. There is a helpful Azerbaijani summary.

R. Bellman (Santa Monica, Calif.).

Mlak, W. Remarks on the stability problem for parabolic equations. Ann. Polon. Math. **3** (1957), 343–348.

The first part of the paper is devoted to proving a stability theorem for nonlinear parabolic differential equations using the generalization of the Westphal-Prodi theorem referred to in the review below. For references to previous work on this problem, see the preceding review of Halilov's paper. The second part of the paper discusses some related stability questions using a result of Szarski [Ann. Polon. Math. **2** (1955), 237–249; MR **17**, 626].

R. Bellman (Santa Monica, Calif.).

Mlak, W. Differential inequalities of parabolic type. Ann. Polon. Math. **3** (1957), 349–354.

The author states and proves some generalizations of the Westphal-Prodi theorem concerning inequalities for non-linear parabolic systems [see G. Prodi, op. cit., review second above]. These results are used in the paper reviewed above to establish stability theorems.

R. Bellman (Santa Monica, Calif.).

See also: Analytic Theory of Numbers: Selberg. **Ordinary Differential Equations:** Krasnosel'skiĭ, Kreĭn and Sobolevskiĭ. **Integral and Integrodifferential Equations:** Bellman. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Aronszajn and Smith. **Manifolds, Connections:** Nakae. **Numerical Methods:** Wasow; Kotal; Juncosa and Young; Douglas.

Difference Equations, Functional Equations

Redheffer, Raymond. On solutions of Riccati's equation as functions of the initial values. J. Rational Mech. Anal. **5** (1956), 835–848.

Les relations

$u_y = a(y)+2b(y)u(x, y)+c(y)u^2(x, y)$, $v_y = b(y)+c(y)u(x, y)$, $w_y = c(y)\exp[2v(x, y)]$, $u(x, x)=0$, $v(x, x)=0$, $w(x, x)=0$ entraînent

$$-w_x = c(x)+2b(x)w(x, y)+a(x)w^2(x, y),$$
$$-v_x = b(x)+a(x)w(x, y),\quad -u_x = a(x)\exp[2v(x, y)],$$

$$(*)\qquad \begin{cases} [u(x, z)-u(y, z)]/u(x, y) = e^{-v(x,y)+v(x,z)+v(y,z)}, \\ 1-u(x, y)w(y, z) = e^{v(x,y)-v(x,z)+v(y,z)}, \\ [w(x, z)-w(x, y)]/w(y, z) = e^{v(x,y)+v(x,z)-v(y,z)}. \end{cases}$$

Les équations fonctionnelles (*) deviennent des identités algébriques relatives aux solutions de l'équation différentielle linéaire du second ordre qui se déduit de l'équation de Riccati en u. Les relations trouvées sont liées avec quelques problèmes de la théorie électromagnetique.

M. Hukuhara (Zbl. **71** (1958), 298).

See also: Sequences, Series, Summability: Hanna. **Partial Differential Equations:** Lavrent'ev.

Integral and Integrodifferential Equations

Grothendieck, A. La théorie de Fredholm. Bull. Soc. Math. France **84** (1956), 319–384.

In the introduction, the author states "Comme l'a vu pour la première fois A. Ruston [Proc. London Math. Soc. (3) **1** (1951), 327–384; MR **13**, 468], le domaine naturel de la théorie de Fredholm se trouve dans les 'opérateurs à trace' de Schatten qui se définissent spontanément dans la théorie des produits tensoriels topologiques. Le présent travail se propose de donner la théorie dans ce cadre général. {The treatment differs from that of Ruston, which had a similar objective.} Bien entendu, on ne saurait, dans un sujet comme la théorie de Fredholm, avoir de prétentions à l'originalité. Aussi cet article ne prétend-il être autre chose qu'un exposé didactique."

In the theory of Fredholm in its classical form, there is an intermediate structure which appears to be dependent on the fact that one has an integral equation, and even the classical expression for the result may seem to depend on integration, i.e., a specialized inner product. The virtue of the present discussion is that the intermediate constructions do not have this "ad hoc" character, but since they are defined in tensorial language they represent procedure having inherent significance. The corresponding logical structure of the discussion is of course much more satisfactory.

F. J. Murray.

Bellman, Richard. Functional equations in the theory of dynamic programming. VII. A partial differential equation for the Fredholm resolvent. Proc. Amer. Math. Soc. **8** (1957), 435–440.

[For other parts, see MR **15**, 887; **17**, 264, 632; **18**, 744; **19**, 428.]. Let $K(x, y)$ be a symmetric kernel over the square $0 \le x, y \le T$, continuous in both variables in this region, and possessing the additional property that $\int_0^T \int_0^T K(x, y)u(x)u(y)dxdy + \int_0^T u^2(x)dx$ is positive definite. Then the Fredholm integral equation

$$u(x)+v(x)+\int_a^T K(x, y)u(y)dy = 0,\quad 0 \le a \le T,$$

has a unique solution for any function $v(x)$ continuous for $a \le x \le T$. This solution may be represented in the form

$$u(x) = -v(x)+\int_a^T Q(x, y, a)v(y)dy.$$

Let us call the kernel $Q(x, y, a)$ the Fredholm resolvent.

The purpose of this note is to show that $Q(x, y, a)$ satisfies the Riccati-type partial differential equation

$$\frac{\partial Q}{\partial a}(x, y, a) = Q(a, x, a)Q(a, y, a).$$

It seems likely that this relation will be of service in connection with the computational solution of linear integral equations, but we shall not discuss this here.

From the introduction.

Hvedelidze, B. V. Singular integral equations with Cauchy kernels in the class of functions that possess weighted sums. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 304–307. (Russian)

1. The author considers the singular integral equation

$$(1)\qquad a(t)\phi(t)+\frac{b(t)}{\pi i}\int\frac{\phi(\tau)}{\tau-t}d\tau = f(t).$$

By the usual method this is reduced to the boundary problem

$$(2)\qquad \Phi^+(t) = \frac{a-b}{a+b}\Phi^-(t)+\frac{f(t)}{a+b}.$$

With respect to the coefficients $a(t)$ and $b(t)$, the absolute term $f(t)$ and the unknown function $\phi(t)$ in (1), such conditions are laid down that the coefficient $(a-b)/(a+b)$, the absolute term $f/(a+b)$ and the unknown function $\Phi(z)$ of the boundary condition (2) belong to the class of functions studied in an article of the author [same Dokl. **111** (1956), 40–43; **MR** 19, 540] under the condition $|\mathrm{Re}\,\lambda_k|<1$. On the basis of the results of that article the author gives a complete solution of equation (1) in the class $L_p(\Gamma, \omega(t))$ of functions in question.

2. For the complete singular equation

$$a(t)\phi(t)+\frac{b(t)}{\pi i}\int\frac{\phi(t)}{\tau-t}d\tau+\int K(t,\tau)\phi(\tau)d\tau=f(t), \tag{3}$$

and also for a system of such equations (with a, b, K matrices and f, ϕ vectors), a regularizing operator is constructed immediately in accordance with equation (3) without recourse to the corresponding boundary problem. Up to the present the construction of a regularizing operator in the case of discontinuous coefficients or boundaries had been possible only on the basis of a solution of a boundary problem.

The results obtained make it possible to prove all the theorems of Noether in the case in question.

F. D. Gahov (RŽMat **1957**, no. 5629).

Čerskiĭ, Yu. I. The general singular equation and equations of convolution type. Mat. Sb. N. S. **41(83)** (1957), 277–296. (Russian)

Singular integral equations of the type

$$a_1(t)\varphi(t)+\frac{a_2(t)}{\pi i}\int\frac{\varphi(\tau)d\tau}{\tau-t}+\int k(t,\tau)\varphi(\tau)d\tau=f(t) \tag{1}$$

were treated by N. I. Mushelišvili [Singular integral equations, OGIZ, Moscow-Leningrad, 1946; MR 8, 586; 15, 434]; a general theory of linear equations was studied by Z. I. Halilov [Linear equations in linear normed spaces, Izdat. Akad. Nauk Azerbaĭdžan SSR, Baku, 1949]. However, Halilov's theory did not cover equations of type (1) adequately. The author generalizes Halilov's work so that the new theory not only encompasses (1) but takes care of the general singular equation

$$M\varphi\equiv A_1\varphi+A_2S\varphi+T\varphi=f, \tag{2}$$

where φ is the solution function, f is an element of a Banach space B, and the operators A_1, A_2, T, S are defined as follows: A_1, A_2 belong to the subring R of the ring Q of all linear (additive and continuous) operators operating on the elements of B; $T\in D\subset Q$, where D is a class of regular operators; $S\in Q$ is a singular operator; "regular" and "singular" are used in the sense defined by Halilov; in addition, it is assumed that the operator $M\equiv A_1+A_2S+T$ is of normal type, i.e., $(A_1+A_2)^{-1}$ and $(A_1-A_2)^{-1}$ exist and belong to R. Under these conditions, (2) is investigated by first considering the Banach spaces B_+, B_-, consisting of the elements φ_+, φ_- of B satisfying $\varphi_+-S\varphi_+=0$ and $\varphi_-+S\varphi_-=0$, respectively. A supplementary problem, analogous to the boundary problem of Riemann, is then stated and solved: to find elements φ_+ and φ_- satisfying (3) $\varphi_+=A\varphi_-+g$, where $g\in B$, $A\in R$ and k is the index of A. Three cases are considered, according as $k<0$, $k=0$, $k>0$. Solving the characteristic equation corresponding to (2), namely, $M^0\varphi\equiv A_1\varphi+A_2S\varphi=f$, is then shown to be equivalent to solving (3), with $A=(A_1+A_2)^{-1}(A_1-A_2)$ and $g=(A_1+A_2)^{-1}f$.

The remainder of the paper is devoted to applying the general theory to singular integral equations of convolution type, less extensive cases of which were considered by the author in previous papers [see, e.g., Gahov and Čerskiĭ, Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 33–52; MR **18**, 134]. Thus he considers, as an instance of (2), operators A of the form

$$A\varphi=\lambda\varphi(x)+(2\pi)^{-1}\int_{-\infty}^{\infty}a(x-t)\varphi(t)dt,$$

where $a(x)\in L_2(-\infty,\infty)$, $\varphi(x)\in L_2(-\infty,\infty)$, T is a completely continuous operator and $S\varphi=\mathrm{sgn}\,x\cdot\varphi(x)$. With additional restrictions on $a(x)$, explicit solutions of (2) are obtained.

J. F. Heyda (Cincinnati, Ohio).

Cernecca, Stellio. Su una classe di equazioni integro-differenziali. Ann. Univ. Ferrara. Sez. VII. (N.S.) 5(1955–1956), 21–47 (1957).

Dans ce travail on donne des conditions nécessaires et suffisantes pour l'existence et l'unicité de solutions de l'équation intégro-différentielle

$$f(P_0)=U(P_0)+\lambda\sum_{j=0}^{M}\sum_{k=0}^{L}\sum_{q_1+\cdots+q_{m_j}=k}\int_{D_j}K^{j|k|q_1\cdots q_{m_j}}(P_0|P_j)\cdot\frac{\partial^k U[\Phi_j(P_j)]}{\partial s_{j_1}^{q_1}\partial s_{j_2}^{q_2}\cdots\partial s_{j_{m_j}}^{q_{m_j}}}dP_j,$$

où P_0 est un point variable dans un domaine carré C_n de l'espace euclidien à n dimensions S_n, $P_j=(s_{j_1},\cdots,s_{j_{m_j}})$ est un point variable dans un domaine rectangulaire D_j de l'espace S_{m_j}, $\Phi_j(P_j)$ est un point de C_n dont les coordonnés sont des fonctions données d'un point P_j de classe L (c'est-a-dire, qui existent et sont continues jusqu'à l'ordre L), $f(P_0)$ et $K^{j|k|q_1\cdots q_{m_j}}(P_0|P_j)$ sont des fonctions données définies respectivement dans les ensembles C_n et $C_n\times D_j$.

Du résumé de l'auteur.

Przeworska-Rolewicz, D. Sur un système d'équations intégrales non-linéaires de seconde espèce à une infinité de fonctions inconnues à singularité forte. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 467–470, XXXIX. (Russian summary)

The author considers an infinite system of nonlinear singular integral equations of the form

$$\varphi_n(t)=\int_L(\tau-t)^{-1}K_n(t,\tau,\varphi_1(\tau),\varphi_2(\tau),\cdots)\,d\tau,$$

where $n=1, 2, \cdots$ and L is a finite set of closed, disjoint, planar curves with continuously turning tangent. The existence of a unique solution is established by the use of a fixed point theorem for continuous transformations in a linear, locally convex, complete metric space [A. Tychonoff, Math. Ann. **111** (1935), 767–776]. The present paper is a generalization of the work of A. I. Guseĭnov [Izv. Akad. Nauk SSSR. Ser. Mat. **12** (1948), 193–212; MR 9, 443].

H. P. Thielman (Ames, Iowa).

See also: Functions of Complex Variables: Vekua. **Trigonometric Series and Integrals:** Michlin. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Vaĭnberg.

Calculus of Variations

Young, L. C. On generalized surfaces of finite topological types. Mem. Amer. Math. Soc. no. **17** (1955), 63 pp.

This important paper presents in virtually complete form a theory of generalized parametric surfaces of finite topological types. From its results follow existence theorems of remarkable generality for 2-dimensional para-

metric problems of the calculus of variations, both regular and non-regular, extending substantially results of Cesari [Amer. J. Math. **74** (1952), 265–295; MR **14**, 292], Danskin [Riv. Mat. Univ. Parma **3** (1952), 43–63; MR **14**, 292], and Sigalov [Uspehi Mat. Nauk (N.S.) **6** (1951), no. 2(42), 16–101; MR **13**, 257, 1139]. Let x denote a point in euclidean m-space and j a skew-symmetric $m\times m$ matrix of rank 0 or 2. R is the unit square in (u, v); $x(u, v)$ a vector-valued generalized Dirichlet representation on R [Young, Rend. Circ. Mat. Palermo (2) **2** (1953), 106–118; MR **15**, 860], and D the class of all such $x(u, v)$; $j(u, v)$ is the vector product of x_u and x_v. Let $C_1, \cdots, C_s$ be disjoint simple closed curves in m-space occupying 2-dimensional measure 0. $x(u, v) \in D$ is said to be of finite topological type with rims $C_1, \cdots, C_s$ if there is a suitable identification process on the perimeter R^* of R such that $C_1, \cdots, C_s$ are represented on the free arcs of R^*. Let F be the space of continuous parametric integrands $f(x, j)$. By parametric surface is meant a linear functional L on F with parametric representation $x(u, v) \in D$:

$$L(f)=\iint_R f[x(u, v), j(u, v)]dudv \text{ (all } f \in F).$$

L is a generalized surface of finite topological type with rims $C_1, \cdots, C_s$ if L is the weak limit of parametric surfaces L_n, where L_n has a representation $x_n(u, v)$ of bounded topological type and the same rims $C_1, \cdots, C_s$ for $n=1, 2, \cdots$. [For non-oriented types F is replaced by the subspace consisting of all f satisfying $f(x, -j)=f(x, j)$.] Closed generalized surfaces of finite topological types are defined similarly. In case $x_n(u, v)$ is constant on R^* for all n, L is termed a generalized sphere. As in the author's previous work on generalized curves [C. R. Soc. Sci. Lett. Varsovie. Cl. III. **30** (1937), 212–234] and generalized nonparametric surfaces [Ann. of Math. (2) **43** (1942), 84–103, 530–544; MR **3**, 249; **4**, 49] the central problem is the existence of a suitable micro-representation M_{uv} for the limit L,

$$L(f)=\iint_R M_{uv}(f)dudv \text{ (all } f \in F)$$

with carrier $x(u, v) \in D$. For almost all (u, v), M_{uv} must be a micro-disc, i.e., a linear average over j-space obtained by concentrating at the point $x(u, v)$ the weak limit of polyhedral discs with planar boundary; the average value of M_{uv} must be $j(u, v)$. The main result of the paper is that if L is of finite topological type, then $L=L_1+L_2$, where L_1 is a generalized sphere and L_2 has a micro-representation of finite topological type with the same rims as L. Conversely, if a generalized surface L has a micro-representation, then L is the weak limit of L_n, where L_n has continuous parametric representation $x_n(u, v)$ such that $x_n(u, v)=x(u, v)$ on R^* for $n=1, 2, \cdots$ ($x(u, v)$ the carrier). The proofs depend on several new tools, of which the following deserve special mention. (A) Sewing theorem: For $i=1, 2$, let L_i be parametric with continuous representation $x_i(u, v) \in D$ satisfying a Lipschitz condition on any compact set interior to R, such that the restrictions $x_1(1, v)$, $x_2(0, v)$ describe for $0\leq v\leq 1$ the same (oriented, Fréchet) curve γ occupying 2-dimensional measure 0; let C_i denote the curve described on the remainder of R^*, $i=1, 2$. Then L_1+L_2 is parametric with continuous representation $x(u, v) \in D$ for which C_1 and C_2 are described on complementary arcs of R^*. (B) An inequality, stated here without proof, between the area of a polyhedron P, minimum length of one-cycles on P not homotopic to 0, and shortest length of a continuum in P whose complement is simply connected. The micro-representation theorem is applied to problems $L(f_0)=\min.$, subject to prescribed topological type and boundary rims. Since not every $x(u, v) \in D$ is continuous, discontinuous solutions might arise. However, under the mild restriction $M(f_0)>0$ for every micro-sphere $M\neq 0$, any solution has a micro-representation with continuous carrier $x(u, v)$. Under weak regularity assumptions on f_0, the carrier itself furnishes a minimum.

W. H. Fleming (Lafayette, Ind.).

Sigalov, A. G. Variational problems with admissible surfaces of arbitrary topological types. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1(73), 53–98. (Russian)

The author extends to surfaces of higher topological types the existence theory, for quasi-regular two-dimensional problems of the calculus of variations in parametric form in Euclidean 3-space, which was previously established for surfaces restricted to the type of the disc [Sigalov, Danskin, Cesari; references cited in review above]. Such a program, without the restriction of quasi-regularity, was carried out earlier by the reviewer in a memoir on generalized surfaces [see the preceding review] except for publication of the details of one lemma, concerning a pinching constant. The reviewer's program likewise extends corresponding work for the type of the disc [Bull. Soc. Math. France **79** (1951), 59–84; MR **13**, 731]. The present extension, like that in the reviewer's memoir, still contains one rather unpleasant restriction in so far as the topological types of admissible surfaces are supposed uniformly bounded, but this restriction has been made until quite recently in all work in this field [see, however, Fleming, Proc. Amer. Math. Soc. **7** (1956), 1063–1074; MR **18**, 489; Young, Comm. Pure Appl. Math. **9** (1956), 625–632; MR **18**, 316].

The fact that the program treated has already been carried out in the reviewer's memoir, does not detract in any way from the importance of the paper under review. In fact, no single proposition is quite the same in the two treatments. This is partly because of slightly different interpretations of the boundary conditions, which necessitate completely different methods of proof, and partly because, while the memoir deals with the more general setting of generalized surfaces, the author of the paper under review has likewise chosen to enlarge, in his own way, the classical setting, by allowing boundary curves which intersect one another and possess multiple points.

The author's treatment makes it necessary for him to introduce a number of definitions (asymptotic degeneracy, strong asymptotic nondegeneracy, asymptotic branching of slits) which concern sequences of polyhedra, or sequences of admissible surfaces, and which are of considerable interest although somewhat difficult to visualize. In other respects, however, the apparatus needed turns out to be remarkably simple, and except for an occasional reference to the author's earlier proof for the case of the disc, it requires no previous knowledge other than standard real variable theory and Chapter VI of Seifert and Threlfall's "Lehrbuch der Topologie" [Teubner, Leipzig-Berlin, 1934]. In particular, the deep theory of surface area is never appealed to. The trick, which makes it possible to bypass sewing theorems and the like and so to avoid using surface area theory, is to adopt a notion of admissible parametric surface, with some defects from the geometrical point of view, as the author observes, but fully adequate for the requirements of quasi-regular variational

problems. This notion is based on Dirichlet representations in canonical domains (the unit square with suitable slits) and on an extremely restricted notion of equivalence (particularly as regards replacing one set of slits by another); actually, the use of any other notion of equivalence would almost inevitably have led to the reviewer's generalized surfaces; and the success of the trick used by the author may be attributed to special circumstances, which have recently been cleared up further by an extension, due to W. H. Fleming [Bull. Amer. Math. Soc. **63** (1957), 32], of Morrey's classical quasi-conformal representation theorem.

The author's existence theorems are based on a lemma for avoiding the asymptotic branching of slits and on an equi-continuity theorem which replaces one in the earlier paper and uses again the reviewer's (ε, δ)-gratings. The existence theorems are stated for three separate types of problems, and each problem splits into two cases, which deal respectively with oriented and non-oriented surfaces. The special case of the problem of least area in its various forms (problem of Plateau, problem of Douglas) is discussed as an application, and analyticity of solution is derived from a quasi-conformal representation theorem.

There is one qualification which the reviewer is compelled to add to the above comments, namely that the author's program be actually capable of being carried out in every detail as outlined by him. In this respect the paper is incomplete, for it is in part based on an elementary error, to which the reviewer's attention was drawn by W. H. Fleming. There can be little doubt that the error can be put right by using the methods of the reviewer's memoir, but this would seem to necessitate appealing to many deep results of area theory, contrary to the whole spirit of Sigalov's approach. The error occurs on pages 88–90, where it is wrongly asserted that a certain elementary change of parameters transforms a representation of the class A^2 into another of the same class. It is hoped that a further note by Sigalov will appear in due course to put this matter right. *L. C. Young.*

See also: Topological Vector Spaces: Borisovič. **Manifolds, Connections:** Bucur. **Elasticity, Plasticity:** Aïnola. **Programming, Resource Allocation, Games:** Bellman.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Braconnier, Jean. L'analyse harmonique dans les groupes abéliens. II. Enseignement Math. (2) **2** (1956), 257–273.

[For Part I see Enseignement Math. (2) **2** (1956), 12–41; MR **18**, 907.] The paper consists of two parts. The first part gives an exposition of the Fourier-Laplace transform as was developed recently by G. Mackey [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 156–162; MR **9**, 497] and J. Riss [Acta Math. **89** (1953), 45–105; MR **14**, 949]. The applications of the Fourier-Laplace transform to the spectral properties of the algebras connected with abelian topological groups are illustrated. The second part gives a compact summary (due to F. Bruhat, I. Gel'fand, R. Godement, Harish-Chandra, G. Mackey, E. Mautner, M. Neumark, I. Segal and others) of the theory of representations of locally compact abelian groups and the algebras connected with them. *K. Yosida* (Tokyo).

Ellis, Robert. Locally compact transformation groups. Duke Math. J. **24** (1957), 119–125.

Let G be a group of homeomorphisms on a locally compact space X. Suppose G has a Hausdorff topology such that multiplication is continuous in each variable separately. If the function $\pi: G\times X\to X$ defined by $\pi(g, x)=g(x)$ is continuous on the left, the author shows that π is jointly continuous. As a consequence, he improves a previous result of his [Proc. Amer. Math. Soc. **8** (1957), 372–373; MR **18**, 745] to the following: If G is a locally compact space admitting a group structure such that multiplication is continuous in each variable separately, then G is a topological group. A generalized Ascoli's theorem for groups is also a consequence. *P. S. Mostert.*

See also: Fields, Rings: Borevič. **Functions of Real Variables:** Katznelson. **Algebraic Topology:** Whitehead; Floyd.

Lie Groups and Algebras

Mostow, G. D. On the fundamental group of a homogeneous space. Ann. of Math. (2) **66** (1957), 249–255.

Let M be a (connected) homogeneous space of a Lie group. The author proves that, if the fundamental group $\pi_1(M)$ is solvable, then $\pi_1(M)$ must be finitely generated and the first Betti number of M cannot exceed the dimension of M. An example is given to show that this theorem no longer holds when $\pi_1(M)$ is not solvable. *H.-C. Wang* (Evanston, Ill.).

See also: Algebras: Albert and Jacobson; Morozov. **Manifolds, Connections:** Kobayashi; Yano.

Topological Vector Spaces

Klee, V. L., Jr. Iteration of the "lin" operation for convex sets. Math. Scand. **4** (1956), 231–238.

If L is a linear space and $X\subseteq L$, then Lin X is the union of X with the set of points of L which are end-points of line segments in L which have their interiors in X. For ordinal numbers α the operations Lin^α are defined by iteration: $\mathrm{Lin}^{\alpha+1} X=\mathrm{Lin}(\mathrm{Lin}^\alpha X)$; and for limit ordinals $\mathrm{Lin}^\alpha X=\bigcup_{\xi<\alpha}\mathrm{Lin}^\xi X$. The order of X is the smallest ordinal α such that $\mathrm{Lin}^{\alpha+1} X=\mathrm{Lin}^\alpha X$. The author gives concise proofs of some results of Nikodým [Rend. Circ. Mat. Palermo (2) **2** (1953), 85–105; **3** (1954), 5–75; MR **15**, 324; **16**, 47] about the possible orders of convex sets in linear spaces of various dimensions. *M. M. Day.*

Sebastião e Silva, J. Le calcul différentiel et intégral dans les espaces localement convexes. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 743–750; **21** (1956), 40–46.

Let B be a family of bounded, convex, circled sets in a locally convex linear topological space X, a family large enough that each point of X is in at least one element of B. If f is a function from an open set D in X into a locally convex linear topological space Y, f is called B-differentiable at a point a in D if there is a linear function L

from X into Y which carries bounded sets of X to bounded sets of Y and is such that $\varphi(x)=f(x)-f(a)-L(x-a)$ is B-infinitesimal of order higher than 1, that is, for each E in B there is a bounded subset C of Y such that for each $\varepsilon>0$ there is a $\delta>0$ such that $\varphi(tx)/t$ is in εC if x is in δB. (When X and Y are normed spaces, the B-differential reduces to the Gateaux-Levy or to the Fréchet differential if B is the set of line segments symmetric about zero or the set of all bounded sets.) The first of these two papers defines also a kind of line integral and generalizes the fundamental theorem of integral calculus. The second paper discusses exact differentials, Taylor's theorem, and analyticity. *M. M. Day* (Urbana, Ill.).

Tomita, Minoru. Compositions of linear topological spaces. Math. J. Okayama Univ. **6** (1957), 191–208.

Let L be a linear topological space and let Λ be a pointwise bounded subset of the set Π of all prenorms (subadditive, absolute-homogeneous functionals) on L. Give Λ and Π the topology of pointwise convergence; then the evaluation mapping $Ef(\lambda)=\lambda(f)$ carries L into $C(\Lambda)$, the space of bounded continuous functions on Λ. The dual mapping $E^*k(f)=k(Ef)$ carries each prenorm k on $C(\Lambda)$ to a prenorm E^*k in Π. For each λ in Λ let $L(\lambda)$ be the quotient space of L by the null-space $\lambda^{-1}(0)$ and set $\|x+\lambda^{-1}(0)\|=\lambda(x)$ to get a normed space. If k is a prenorm on $C(\Lambda)$, then E^*k is called a composite of the prenorms in Λ and $L(E^*k)$ is called a composite of the spaces $L(\lambda)$, λ in Λ. The author considers also the effect of this composition on the conjugate spaces of the $L(\lambda)$ spaces, and, finally, applies this composition to the theory of decomposition of operator algebras. *M. M. Day* (Urbana, Ill.).

Bessaga, C.; and Pełczyński, A. On a class of B_0-spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 375–377, XXX. (Russian summary)

The class of B_0-spaces in the sense of S. Mazur and W. Orlicz [Studia Math. **10** (1948), 184–208; MR **10**, 611] coincides with the class of complete metrizable locally convex topological vector spaces or, what is the same, the Fréchet spaces in the terminology of N. Bourbaki [Espaces vectoriels topologiques, Ch. I, II, Actualités Sci. Ind., no. 1189, Hermann, Paris, 1953, p. 59; MR **14**, 880]. The present paper is concerned with the subclass $\mathfrak{R}$ of all those B_0-spaces in which there exists no continuous homogeneous norm (here the terms "norm", "pseudonorm" are used in the same sense as in the paper of Mazur-Orlicz). This class $\mathfrak{R}$ arises in the authors' generalization [see the paper reviewed below] of a theorem of Krein-Milman-Rutman concerning bases. The following results are proved. A B_0-space X with homogeneous pseudonorms $\|\cdot\|_k$ $(k=1, 2, 3, \cdots)$ is in $\mathfrak{R}$ if and only if, for any finite n, there exists an $x\in X$ such that $x\neq 0$ and $\|x\|_k=0$ for $1\leq k\leq n$. Let s denote the B_0-space of all real sequences $x=\{\xi_n\}$ with pseudonorms $\|x\|_k=|\xi_k|$ $(k=1, 2, 3, \cdots)$. Then a B_0-space X is in $\mathfrak{R}$ if and only if $X=Y\times Z$, where Y is a B_0-space and Z is a space isomorphic to s. For a B_0-space X with pseudonorms $\|\cdot\|_k$ $(k=1, 2, 3, \cdots)$, let X_k denote the quotient space $X/\{x\in X:\|x\|_k=0\}$ with the norm induced by $\|\cdot\|_k$. Then an infinite-dimensional B_0-space X is isomorphic to s if and only if the dimension of each X_k is finite. *Ky Fan.*

Bessaga, C.; and Pełczyński, A. An extension of the Krein-Milman-Rutman theorem concerning bases to the case of B_0-spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 379–383, XXX. (Russian summary)

The Krein-Milman-Rutman theorem [Zap. Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) **16** (1940), 106–110; MR **3**, 49] states that, if a Banach space X has a basis, then every dense subset of X contains a basis of X. In the present paper, the following generalization is proved for complete metrizable locally convex topological vector spaces or, what is the same, B_0-spaces in the terminology of Mazur and Orlicz [reference in review above]: If a B_0-space X has a basis, then every dense subset of X contains a basis of X if and only if there exists a continuous homogeneous norm on X, i.e., X is not in the class $\mathfrak{R}$ discussed in the paper reviewed above. Let s denote the B_0-space of all real sequences $x=\{\xi_n\}$ with pseudonorms $\|x\|_k=|\xi_k|$ $(k=1, 2, 3, \cdots)$. It is shown that a B_0-space X of the class $\mathfrak{R}$ is isomorphic to s if and only if every dense linear subset of X contains a basis of X. *Ky Fan* (Oak Ridge, Tenn.).

Rolewicz, S. On a certain class of linear metric spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 471–473, XL. (Russian summary)

Let X be a metrizable topological vector space. Let $\mathfrak{A}$ denote the family of all bounded neighborhoods A of 0 in X such that $tA\subset A$ for $|t|\leq 1$. The concavity module $c(A)$ of a set $A\in\mathfrak{A}$ is defined to be the infimum of all positive t satisfying $A+A\subset tA$. The infimum $c(X)$ of all $c(A)$, $A\in\mathfrak{A}$, is called the concavity module of the space X, and is always ≥ 2. The following theorem is proved. If $0<p<p_0=\log_{c(X)}2$, then the topology of X can be defined by a p-homogeneous norm, i.e., a function $\|\cdot\|$ which has all the usual properties of a norm, except that the homogeneity condition is replaced by $\|tx\|=|t|^p\|x\|$. The paper includes an example (due to A. Pełczyński) showing that the hypothesis $p<p_0$ in the theorem is essential. Also the concavity modules for both spaces l^p and $L^p(0, 1)$, where $0<p<1$, are determined to be $2^{1/p}$. *Ky Fan.*

Pavel, M. On quasi normed spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 479–484, XL. (Russian summary)

An r-quasi-norm for $0<r\leq 1$ is a non-negative valued function $\|\cdot\|$ on a linear space, which vanishes only at zero, satisfies the triangle inequality, and satisfies the relation $\|\lambda x\|=|\lambda|^r\|x\|$ for all x in L and all scalars λ. A complete linear topological space whose topology is given by some r-quasinorm is called a (QN) space. The author discusses briefly finite products of (QN) spaces, quasinorms for linear operators between (QN) spaces, resolvent and spectrum in (QN) algebras, and holomorphic functions in (QN) algebras. *M. M. Day* (Urbana, Ill.).

Weston, J. D. A topological characterization of L-spaces. J. London Math. Soc. **32** (1957), 473–476.

The author proves that, if X is a Hausdorff linear topological space which is a vector lattice, then the following conditions are together necessary and sufficient that the given topology be determined by a norm in which the space becomes an abstract L-space: (a) The absolute value of x goes to zero with x in X. (b) There is a continuous linear functional f on X such that $\{x: x\geq 0$ and $f(x)=1\}$ is bounded and non-empty. (c) X is complete. (d) The positive cone in X is closed. *M. M. Day.*

Weston, J. D. Convergence of monotonic sequences in vector spaces. J. London Math. Soc. **32** (1957), 476–477.

This note contains a brief proof of the following result of Bonsall [Proc. Durham Philos. Soc. Sect. A **13** (1957),

6–11]: Let X be a partially-ordered locally-convex linear topological space in which for each neighborhood G of 0 there is a neighborhood H of 0 such that G contains all the intervals $\{z: x\leqq z\leqq y\}$ with both ends x, y in H: let (x_n) be a sequence weakly convergent to 0 in X such that for all n, $x_n\geqq x_{n+1}\geqq 0$; then x_n converges to 0 in X.

M. M. Day (Urbana, Ill.).

Amemiya, Ichiro; and Mori, Tuyosi. Topological structures in ordered linear spaces. J. Math. Soc. Japan 9 (1957), 131–142.

L denotes always a conditionally complete lattice. T_0 denotes order topology, i.e., a set is T_0-closed if and only if it is closed under directed order convergence. In general, lattice operations are not T_0-continuous. Now, a separated topology on L is called a T_t topology if: (C_1) T_t is weaker than T_0, (C_2) the lattice operations are T_t-continuous, and (C_3) every x in L has a basis of T_t neighborhoods which are order-closed and order-convex. (If $L=B$ is a Boolean algebra with a countably additive non-negative measure $\mu(x)$ such that $\mu(x)=0$ only if $x=0$, then convergence in measure determines such a T_t topology.)

If L is complete, the T_t topology, if it exists at all, is unique; thus, in the conditionally complete case, on every interval the T_t topology is unique. If $L=E$ is a vector lattice or a reducible continuous geometry, then L has a T_t topology if and only if the Boolean algebra of its direct summands has a T_t topology.

A complete Boolean algebra B has a compact T_t topology if and only if it is atomic, i.e., B is isomorphic to the lattice of all subsets of a fixed set. Then the T_t topology coincides with the order topology. This theorem is applied to give a new proof for the theorem: if E is a conditionally complete vector lattice or reducible continuous geometry, then E has a T_t topology under which every interval is compact if and only if the Boolean algebra of all its direct summands is atomic [this reformulates a theorem of Halperin and Nakano, Canad. J. Math. 3 (1951), 293–298; MR 13, 49]; and to prove the theorem: if the unit sphere of a ring of operators is compact under the strong topology then it is the (discrete) direct sum of factors of type I_n.

If B is a Boolean algebra with T_t topology then B is actually a topological abelian group under the operation of symmetric difference; if E is a vector lattice with T_t topology then there exists some T_t topology which is also a linear topology, and in fact, there exists a weakest such topology. Thus, in these cases, the T_t topology is induced by a uniform structure. The fact that this uniform structure is complete is part of the general theorem: a uniform structure $\mathfrak{U}$ on a conditionally complete L is complete if: (i) $\mathfrak{U}$ has a T_0-closed basis in $L\times L$, (ii) the lattice operations are uniformly continuous on $L\times L$, and (iii) every Cauchy monotone family in L is convergent [this generalizes a theorem of Nakano for vector lattices, J. Fac. Sci. Hokkaido Univ. Ser. I. 12 (1953), 87–104; MR 15, 137].

If E' is total on E (vector lattice) then the topology $\beta(E, E')$ is convex-linear, satisfies (C_2) and (C_3) and is the strongest such topology. Also, in this case, the convex-linear topologies $\tau(E, E')$ and $\tau(E', E)$ both satisfy (C_2) and (C_3) and hence are T_t topologies. This last theorem, given explicitly for the first time, is proved by a new method but, as the authors point out, is closely related to ideas used by Dunford-Pettis, Nakano, and J. L. B. Cooper.

The roles of (C_2) and (C_3) are clarified by interesting examples.

{This review was written in consultation with one of the authors, I. Amemiya.} *I. Halperin.*

Cotlar, M. A general interpolation theorem for linear operations. Rev. Mat. Cuyana 1 (1955), 57–84 (1956).

Let T be a sublinear operator defined on the space D of real, finitely many-valued, step functions on R^n. It is of type p if $\|Tf\|_p\leqq M\|f\|_p$ for all $f\in D$. If T is linear, it can be extended to a bounded operator on L^p. A consequence of M. Riesz's convexity theorem is that, if T is of types p_1 and p_2, $1\leqq p_1<p_2$, then it is also of type p for every p in (p_1, p_2). The author shows that this conclusion is valid under a weaker hypothesis, namely, if $\|Tf\|_{p_1}$ and $\|Tf\|_{p_2}$ are replaced by various smaller values.

Let $S(f)$ be the support of f, and $m(f)$ and $M(f)$ the minimum and maximum respectively of $|f(x)|$ on $S(f)$. $P_m f(x)=m(f)$ if $|Tf(x)|\geqq m(f)$ and is zero otherwise; $P_M f(x)=M(f)$ if $|Tf(x)|\geqq M(f)$ and is zero otherwise. T is of m-type p if $\|P_m f\|_p\leqq A_p\|f\|_p$; M-type p is analogously defined. A_p denotes a constant depending on p alone. Let E be the set of points on which $|Tf|\geqq\lambda$ and $|E|$ be its measure. T is of m-type p if and only if $|E|\leqq A_p\|f/\lambda\|_p{}^p$ for every $\lambda\leqq m(f)$. T is of M-type p if and only if this inequality holds for every $\lambda\geqq M(f)$. If T satisfies this inequality for all $\lambda>0$, it is said to satisfy condition (p).

The author shows that if T is of m-type p_1 and M-type p_2, then T is of type p, $p_1<p<p_2$. Furthermore, T is of m-type p_1 and M-type p_2 if and only if $\int_S|Tf|^\alpha dx\leqq A_p(p-\alpha)^{-1}|S|^{1-\alpha/p}\|f\|_p{}^\alpha$ for every set S of finite measure and α, p such that $0<\alpha<p_1$, $p_1\leqq p\leqq p_2$.

In the remaining portion of the paper the author takes $p_1=1$, $p_2=2$, and T satisfying condition (2).

Let L be any fixed sublinear operator on D of m-type 1 and M-type 2. $S_L(f)=\{x: |Lf(x)|\geqq m(f)\}$. For each L, each $f\in D$, and each $\Delta>0$, define $V(f, L, \Delta)$ to be the set of all step functions $\delta(x)$ which vanish outside $S_L(f)$ and are always less in absolute value than $\Delta m(f)$. The "modified norm" $\|Tf\|_{L,\Delta}=\inf_{\delta\in V(f,L,\Delta)}\{\int_{R^n-S_L(f)}|T[f(x)-\delta(x)]|dx\}$. The author shows that if $\|Tf\|_{L,\Delta}\leqq A_1\|f\|_1$, for some L and Δ, and for all $f\in D$, then T is of m-type 1, and hence of type p for every p, $1<p<2$.

Let $Q(x_0)$ be a cube with center x_0 and edges parallel to the axes of R^n; $\frac{1}{2}Q$ is the cube with edge length $\frac{1}{2}$ that of Q; ϕ_E denotes the characteristic function of the set E. The author defines

$$\|Tf\|_{L,\Delta,Q}=\sup_{\frac{1}{2}Q\subset K\subset Q}\inf_{\delta\in V}\int_{R^n-S_L(f)}|T[\phi_E(x)f(x)-\phi_Q(x)f(x)]|dx,$$

and shows that

$$\|Tf\|_{L,\Delta}=\sup_{f\in D}\lim_{Q(x_0)\to\infty}\frac{\|Tf\|_{L,\Delta,Q(x_0)}}{\|f\phi_Q\|_1}.$$

Finally, he defines

$$\|T\|_{L,\Delta,x_0}=\sup_{f\in D}\inf_{Q(x_0)}\frac{\|Tf\|_{L,\Delta,Q}}{\|f\phi_Q\|_1}.$$

The main result is this: If Δ and L are as above, T satisfies condition (2), and $\|T\|_{L,\Delta,x_0}\leqq A$ for all x_0 in R^n, then $\|T\|_{L,2^n\Delta}\leqq 2^n A$. Also, T is of m-type 1 and of type p for every p, $1<p<2$.

Several other results are given which are more special and more readily applicable. They are used to prove several results on singular integrals previously established by M. Riesz, Kolmogoroff, and Zygmund and Calderón.

There are certain connections with a paper of Zygmund [J. Math. Pures Appl. (9) **35** (1956), 223–248; MR **18**, 321]. *D. Waterman* (Lafayette, Ind.).

Cotlar, M. Some generalizations of the Hardy-Littlewood maximal theorem. Rev. Mat. Cuyana **1** (1955), 85–104 (1956). (Spanish summary)

Soit $R^n=\{x\}$ un espace euclidien, $D=\{f(x)\}$ une classe de fonctions dense dans $L^p\{R^n\}$, et $T=Tf$ un opérateur défini dans D, tel que (1) $|T(f+g)|\leqq|Tf|+|Tg|$, $f, g\in D$. D'après la note analysée ci-dessus, T sera du type p si sa norme satisfait à

$$\|Tf\|_p{}^p=\int_{R^n}|Tf|^p dx\leqq O_p\int_{R^n}|f|^p dx,$$

pour tout $f\in D$. T satisfait la condition (p) si

$$|E[|Tf|\geqq\lambda]|\leqq O_p\lambda^{-p}\int_{R^n}|f|^p dx,$$

pour tout $\lambda>0$, $f\in D$, $|E[\cdot]|$ désignant la mesure de $E[\cdot]$. Soit $M=Mf$ un autre opérateur défini dans R^n satisfaisant à (1), alors $|M|^\alpha\prec O_\alpha|T|^\alpha$ signifie que pour tout $f\in D$, $x\in R^n$, il existe un cube à n dimensions, avec le centre dans x et tel que $|Mf(x)|^\alpha\leqq O_\alpha|Q(x)|^{-1}\int_{Q(x)}\varphi_{Q(x)}(t)|Tf(t)|^\alpha dt$, $\varphi_Q(t)$ étant la fonction caractéristique d'ensemble Q. $|M^\alpha<<O_\alpha|T|^\alpha$ signifie que $|Mf(x)|^\alpha\leqq O_\alpha|Q|^{-1}\cdot\int_Q|T(\varphi_{Q(x)}f(t))|^\alpha dt$. Le théorème maximal de Hardy et Littlewood [Zygmund, Trigonometrical series, Warsaw, 1935, p. 244] correspond alors au cas (2) $|M|\prec O_1|I|$, où I désigne l'opérateur identique. Pour les opérateurs ainsi définis l'auteur a obtenu des résultats générales sous la forme suivante. Si $|M|^\alpha\prec O_\alpha\cdot|T|^\alpha$ et T satisfait à (p) (respect. si T est du type p), $p>\alpha$, M satisfait aussi à (p) (respect. M est du type p). Si $|M|^\alpha<<|T|^\alpha$ et T satisfait à (p), $p>\alpha$, M satisfait aussi à (p) et est du type q, $q>p$. Dans les cas spéciales $R^n=R^1$, $R^n=R^2$ de ces résultats découlent quelques théorèmes connus obtenus par une voie différente [pour R^1: Zygmund, loc. cit. § 10.22; pour R^2: Calderón et Zygmund, Acta Math. **88** (1952), 85–139; MR **14**, 637]. On en déduit ensuite des résultats analogues pour les produits des opérateurs. L'auteur a établi un théorème analogue au théorème maximal de Hardy et Littlewood pour la transformation double définie dans $R^{n+m}=R^n\times R^m$, avec une démonstration directe. {Remarque: Une généralisation de l'inégalité (2) dans l'espace de Banach, avec les spécialisations dans l'espace euclidien à n dimensions est énoncée aussi par K. T. Smith [Bull. Amer. Math. Soc. **60** (1954), 546].} *M. Tomić*.

Wiweger, A. A topologisation of Saks spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 773–777, LXVII. (Russian summary)

Let τ and τ^* be linear space topologies on L subject to the restriction that τ has at 0 a neighborhood basis $\mathfrak{U}$ of convex, symmetric, τ^*-closed sets. A topology $\bar{\tau}$ is defined by choosing as a neighborhood basis at 0 all sets of the form $\bigcup_{n<\infty}(U_1{}^*\cap U+U_2{}^*\cap 2U+\cdots+U_n{}^*\cap nU)$, where U is in $\mathfrak{U}$ and all $U_i{}^*$ are τ^*-neighborhoods of 0. The $\bar{\tau}$ topology is stronger than τ^* and agrees with the latter on every τ-bounded set; if τ is a norm topology, then $\bar{\tau}$ is the strongest topology agreeing with τ^* on all τ-bounded sets. If, in this case, U is a ball of the τ-norm about 0, then $\bar{\tau}$ is also the weakest topology τ' such that every linear function whose restriction to U is τ^*-continuous on U is also τ'-continuous on L.

The construction of the paper was chosen in order to topologize the "two-norm convergence" [Alexiewicz, Studia Math. **14** (1953), 49–56; MR **15**, 880]; this construction would also apply to the bounded weak and bounded weak* topologies of a normed space [Dieudonné, Proc. Amer. Math. Soc. **1** (1950), 54–59; MR **11**, 524]. *M. M. Day* (Urbana, Ill.).

Borisovič, Yu. G. On stability of the critical values of even functionals. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 165–168. (Russian)

The author generalizes the definition [Krasnosel'skiĭ, Uspehi Mat. Nauk (N.S.) **7** (1952), no. 2(48), 157–164; MR **14**, 55] of the genus of a set in a metric space R as follows. Let A be a continuous mapping of a metric space R into itself. If $E\subset R$ is closed, its genus relative to the mapping A is 1 if no component of E contains a point and its image under A. The genus is n if E is a subset of the union of n sets of genus 1 and is not a subset of any union of $n-1$ sets of genus 1. The genus has many of the properties of the Lyusternik-Šnirel'man category. The author proves the following theorem. Let $F(\varphi)$ and $G(\varphi)$ be functionals on the linear normed space R, with $F(A\varphi)=F(\varphi)$ and $G(A\varphi)=G(\varphi)$ [in Krasnosel'skiĭ's paper A was an involution]. Let F be weakly continuous. If R is a Banach space suppose $A^2=1$; if R is a Hilbert space suppose $A^*A=1$. Two points of R are essentially different if neither can be gotten from the other by repeated applications of A. Then the number of essentially different points φ for which there is a λ with grad $F(\varphi)-\lambda$ grad $G(\varphi)=0$, i.e., critical points, on the surface $G(\varphi)=k$, is at least equal to the genus of the surface $G(\varphi)=k$ with respect to A. *J. M. Danskin* (Princeton, N.J.).

See also: Linear Algebra: Dean. **Algebras:** Biatynicki-Birula and Żelazko. **Measure, Integration:** Mařík. **Ordinary Differential Equations:** Krasnosel'skiĭ, Kreĭn, and Sobolevskiĭ. **Partial Differential Equations:** Trèves; Shirota; Sobolev. **Integral and Integrodifferential Equations:** Čerskiĭ; Przeworska-Rolewicz. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Matuszewska and Orlicz; Lumer; Dinculeanu; Zuhovickiĭ; Smart; Aronszajn and Smith; Pachale.

Banach Spaces, Banach Algebras, Hilbert Spaces

Matuszewska, W.; and Orlicz, W. On a class of Saks spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 611–614, LII. (Russian summary)

Let $\varphi(u)$ be a convex and continuous function for $u\geqq 0$ satisfying the following conditions: a) $\varphi(0)=0$, $\varphi(u)>0$ for $u>0$; b) $\varphi(u)/u\to 0$ for $u\to 0$; c) $\varphi(u)/u\to\infty$ for $u\to\infty$. $\psi(u)$ is the function complementary to $\varphi(u)$. $L^{\varphi\psi}$ is the set of all real-valued measurable functions $x(t)$ in $(-\infty, +\infty)$ for which $\Im_\varphi(k\cdot x)=\int_{-\infty}^{+\infty}\varphi(k\cdot x(t))dt$ is finite for a certain $k>0$. $M^{\varphi\psi}$ is the subclass of $L^{\varphi\psi}$ consisting of the essentially bounded functions in $(-\infty, +\infty)$. In $M^{\varphi\psi}$ the following norms are introduced: $\|x\|=\text{ess.sup}|x(t)|$; $\|x\|^*=\inf(l;\Im_\psi(x/l)\leqq 1, l>0)$; $\|x\|^0=\sup(\int_{-\infty}^{+\infty}|x(t)\cdot y(t)|dt;\Im_\psi(y)\leqq 1)$. Then, as is well known, $\|x\|^*\leqq\|x\|^0\leqq 2\|x\|^*$. $M_s^{\varphi\psi}=\{x; x\in M^{\varphi\psi}, \|x\|\leqq 1\}$, with the metric $d(x, y)=\|x-y\|^*$, is a Saks space. In this paper the authors prove among other things that $M_s^{\varphi\psi}$ is separable if and only if $\lim\sup_{u\to 0}\varphi(2u)/\varphi(u)$ is finite. A distributive functional ξ on $M^{\varphi\psi}$ is called linear on $M_s^{\varphi\psi}$ if $\|x_n\|\leqq 1$, and $\|x_n\|^*\to 0$ implies $\xi(x_n)\to 0$. A general representation theorem for such functionals is given. *W. A. J. Luxemburg* (Toronto, Ont.).

Bogdanowicz, W. On the weak continuity of the polynomial functionals defined on the space c_0. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 243–246, XXI. (Russian. English summary)

Let m^* denote the space of bounded sequences, with its weak* topology of the adjoint of l. It is proved that every continuous (with respect to the norm topology) k-linear form on c_0 (the space of sequences converging to 0) admits of a (unique) extension to a continuous k-linear form on m^*. *M. Katětov* (Prague).

Pełczyński, A. On B-spaces containing subspaces isomorphic to the space c_0. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 797–798, LXVIII. (Russian summary)

A sequence is unconditionally summable if its partial sums converge for any rearrangement of the sequence. A B-space contains a weakly unconditionally summable sequence which is not strongly unconditionally summable if and only if it contains a subspace isomorphic to c_0. Several consequences of this theorem are mentioned. The proofs will appear in Studia Mathematica.

D. C. Kleinecke (Livermore, Calif.).

Ruston, A. F. A note on clorms. J. London Math. Soc. **32** (1957), 110–112.

The author presents a generalization of his previous proof of certain results of Schatten on self-associate cross norms [Proc. London Math. Soc. (3) **1** (1951), 327–384; MR **13**, 468]. He removes the restriction to reflexive Banach space by introducing a property "complete uniformity" of the "clorm" or norm on the cross space.

F. J. Murray (New York, N.Y.).

Huhunaišvili, G. E. Continuous images of Hilbert space. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 275–288. (Russian)

Necessary and sufficient conditions are found which must be satisfied by a metric space, in order that it may be the continuous image of a Hilbert space.

Author's summary.

Lumer, G. The range of the exponential function. Bol. Fac. Ingen. Agrimens. Montevideo **6**=Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist. **3** (1957), 53–55.

Let H be an infinite-dimensional Hilbert space, $E[H]$ the algebra of linear bounded operators on H to itself. Let G be the set of regular elements of $E[H]$, $G_0=K$ the maximal connected subset of G containing the identity, and let R_2 be the set of elements having square roots in $E[H]$. P. Halmos, G. Lumer, and J. J. Schäffer [Proc. Amer. Math. Soc. **4** (1953), 142–149; MR **14**, 767] exhibited a subset of $G_0=G$ not in R_2 and later [ibid. **5** (1954), 589–595; MR **16**, 48] Halmos and Lumer showed that $G \ominus R_2$ has interior points. The author now points out that the range R of the exponential function in $E[H]$ is consequently a proper subset of G and R is not dense in G. (See also A. Wintner, Amer. J. Math. **79** (1957), 139–151 [MR **18**, 483], who drew similar conclusions from the results of these authors and who pointed out that R is not a group.) *E. Hille* (New Haven, Conn.).

Lumer, G. Commutators in Banach algebras. Bol. Fac. Ingen. Agrimens. Montevideo **6**=Fac. Ingen. Agrimens. Montevideo. Publ. Inst. Mat. Estadist. **3** (1957), 57–63. (Spanish. English summary)

If B is a Banach algebra with identity e and $c=ab-ba$ ($a, b \in B$) is a commutator element in B, then $\|c-e\| \geqq 1$ in many cases (but not, for example, for all commutators in an infinite-dimensional Hilbert Space). Several conditions are given for the inequality, including: if c commutes with ab, or if a is a limit of operators reduced by eigen-spaces, or if B contains an invariant commutative subalgebra with unit. *D. C. Kleinecke* (Livermore, Calif.).

Putnam, C. R. On square roots of normal operators. Proc. Amer. Math. Soc. **8** (1957), 768–769.

Let N be a normal operator on a Hilbert space $\mathfrak{H}$ and let A be such that $A^2=N$. The author proves that, if the convex set W consisting of all complex numbers of the form (Ax, x), $\|x\|=1$, lies entirely on or on one side of a straight line in the complex plane which passes through the origin, then A is normal. If N is self-adjoint and $A^2=N$, then, if the set W consists of numbers with non-zero real parts, A is also self-adjoint. If all elements of W have non-zero imaginary parts, iA is self-adjoint.

R. E. Fullerton (College Park, Md.).

Butler, John. On the normality of an analytic operator. Proc. Amer. Math. Soc. **8** (1957), 733–740.

Let $T(z)$ be a family of closed operators on a Hilbert space $\mathfrak{H}$ depending on the complex parameter z. The principal theorem states that, if the family $T(z)$ satisfies (a) $T(z)$, $T^*(z)$ have fixed domains $\mathfrak{D}$ for $|z|<\gamma$, (b) $T(z)f$, $T^*(\bar{z})f$ are analytic in z, $|z|<\gamma$, $f \in \mathfrak{D}$, (c) $T(0)$ is normal, then in some neighborhood of the origin, the set of points z for which $T(z)$ is normal either (i) reduces to a single point, (ii) fills a neighborhood of the origin, or (iii) consists of a finite number of analytic arcs. As an application the author gives the following generalization of a theorem of Rellich [Math. Ann. **116** (1939), 555–570; **118** (1942), 462–484; MR **6**, 71]. If $T(z)$ satisfies (a), (b), (c) and in addition if $T(x_n)$ is normal on a sequence of real positive numbers $\{x_n\}$, $\lim_{n\to\infty} x_n=0$, then there exist $\delta>0$ and a set of functions $\{\mu_i(x)\}$ analytic for $0 \leqq x<\delta$ such that for $0 \leqq x<\delta$ the set of spectral values of the $T(x)$ is the set of numbers $\{\mu_i(x)\}$. *R. E. Fullerton.*

Thoma, Elmar. Zur Reduktionstheorie in separablen Hilbert-Räumen. Math. Z. **67** (1957), 1–9.

Let X be a set, $\mathfrak{R}$ a σ-field of subsets of X and μ a completely additive σ-finite measure on $\mathfrak{R}$. To each $x \in X$ associate a separable Hilbert space $\mathfrak{H}_x$ and let f be a function from X to $\mathbf{U}_{x \in X} \mathfrak{H}_x$ with $f(x) \in \mathfrak{H}_x$ for each $x \in X$. Let $\mathfrak{M}$ be a maximal set of such functions such that $f, g \in \mathfrak{M}$ implies $(g(x), f(x))$ summable $(\mu, \mathfrak{R})$. If $\mathfrak{M}$ is given an inner product $(f, g)=\int_X (f(x), g(x))d\mu$ and if identification of functions differing on null sets is made, the space $\mathfrak{S}$ with this inner product is a Hilbert space called a von Neumann direct integral of the spaces $\{\mathfrak{H}_x\}$. The author proves an existence and uniqueness theorem for representing separable Hilbert spaces as direct integrals. If $\mathfrak{H}$ is a separable Hilbert space, X, $\mathfrak{R}$, μ as above, and E a spectral measure on $\mathfrak{H}$ with domain $\mathfrak{R}$, (i.e., E is a family of projections with $E(X)=I$ and E completely additive over $\mathfrak{R}$), then $\mathfrak{H}$ is unitarily equivalent to a direct integral of Hilbert spaces defined over X, with E equivalent to a canonical spectral measure E', where $E'(M)f=\chi_M f$, χ_M the characteristic function of M. The uniqueness proof shows that if $\mathfrak{S}_1$, $\mathfrak{S}_2$ are two different representations as direct integrals then almost all component spaces are equivalent under unitary transformations and the measure functions on X which define the direct integrals are quivalent.

R. E. Fullerton (College Park, Md.).

Dinculeanu, Nicolae. Espaces d'Orlicz de champs de vecteurs. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 135–139.

Let Z be a locally compact Hausdorff space and μ a positive Radon measure on Z. Let $E=(E(z); z\in Z)$ be a family of Banach spaces and $C(E)$ the class of all functions $\mathbf{x}(z)$ defined on Z with $\mathbf{x}(z)\in E(z)$. Under the assumption that there exists a fundamental family $ACC(E)$ of continuous functions, R. Godement [C. R. Acad. Sci. Paris **228** (1949), 1321–1323; MR **10**, 548] has introduced the spaces L_A^p of all functions $\mathbf{x}(z)$ of which the function $x(z)=\|\mathbf{x}(z)\|^p$ is μ-integrable. In this paper the author deals with a generalization of Godement's theory. He replaces the p-integrability by Φ-integrability, where $\Phi(u)$ is some non-negative increasing and left continuous function defined for $0\leqq u<\infty$ with $\Phi(0)=0$ and $\lim_{u\to\infty}\Phi(u)=\infty$. *W. A. J. Luxemburg* (Toronto, Ont.).

Zuhovickiĭ, S. I. On minimal extensions of linear functionals in a space of continuous functions. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 409–422. (Russian)

Let Q be a locally compact Hausdorff space, and let $C_\infty(Q)$ be the normed linear space of all complex-valued continuous functions on Q that vanish at ∞, with the uniform norm. Let G be a linear subspace of $C_\infty(Q)$, and f a bounded complex linear functional on G such that $\|f\|=f(\phi)$ for some $\phi\in G$, $\|\phi\|=1$. Let $m(\psi)=\int_Q \psi(x)d\mu(x)$ be an extension of f over all of $C_\infty(Q)$, written as the integral with respect to a countably additive, regular, complex-valued, finite, Borel measure on Q. Then $\|m\|=\|f\|$ if and only if $d\mu(x)=\Phi(x)d|\mu|(x)$. Furthermore, let $m(\psi)=\int_Q \psi(x)d\mu(x)$ be a bounded linear functional on $C_\infty(Q)$. Then there is a $\phi\in C_\infty(Q)$ such that $\|\phi\|=1$ and $m(\phi)=\|m\|$ if and only if $d\mu(x)=\alpha(x)d|\mu|(x)$, where α is a continuous complex-valued function on Q. The author proves these theorems for the cases Q a closed bounded interval on the real line and Q an arbitrary compact metric space. The proofs carry over at once to the more general situation described above. Analogous results hold for spaces of real-valued continuous functions. [See also W. W. Rogosinski, Proc. London Math. Soc. (3) **6** (1956), 175–190; MR **17**, 987.] *E. Hewitt* (Seattle, Wash.).

Smart, D. R. Representation of Hilbert space operators by (nJ)-matrices. Proc. Cambridge Philos. Soc. **53** (1957), 304–311.

Let $\mathfrak{H}$ be a separable complex Hilbert space and let H be an infinite matrix which, with respect to some complete orthonormal set, represents a linear operator T. H is an (nJ) matrix if it is Hermitian and if $H=\{h_{ij}\}$, $h_{ij}=0$ for $i-j>n$ and $h_{ij}\neq 0$ for $i-j=n$. If in addition $h_{ij}=0$ for $0<i-j<n$, H is called an $(nJ\perp)$ matrix. The two principal results of this paper are generalizations of results of Stone [Linear transformations in Hilbert space, Amer. Math. Soc. Colloq. Publ., v. 15, New York, 1932] on operators represented by Jacobi matrices ($(2J)$ matrices). The first of these theorems states that a necessary and sufficient condition that a Hermitian linear operator T be representable by an (nJ) matrix is that T have an n-dimensional generating subspace, where $K\subset\mathfrak{H}$ is an n-dimensional generating subspace if K is n-dimensional and $K, TK, T^2K, T^3K, \cdots$ are linearly independent of one another and generate $\mathfrak{H}$. The second principal theorem states (a) that if T can be represented by an (nJ) matrix, there exists $p\leqq n$ with T representable by a $(pJ\perp)$ matrix, (b) if T is representable by a $(pJ\perp)$ matrix, every spectral point of T has multiplicity not exceeding p, and (c) if every spectral point of T has multiplicity not exceeding q, then T is representable by an $(rJ\perp)$ matrix for any $r\geqq q$. *R. E. Fullerton* (College Park, Md.).

Halberg, Charles J. A., Jr. The spectra of bounded linear operators on the sequence spaces. Proc. Amer. Math. Soc. **8** (1957), 728–732.

Let $1\leqq p\leqq q\leqq r\leqq\infty$ and let T be simultaneously a bounded linear operator on l_p and l_r. The convexity theorem of M. Riesz [Acta Math. **49** (1927), 465–497] implies that T is also a bounded operator on l_q. By elementary arguments based on the Riesz theorem the author derives various relationships between the spectra and spectral radii of T considered as an operator on these spaces. Thus if $\sigma(T_p)$, $|\sigma(T_p)|$ etc., denote these spectra and spectral radii respectively, it is shown that $\sigma(T_q)\subset\sigma(T_p)\cup\sigma(T_r)$, that $\log|\sigma(T_{1/\alpha})|$ is a convex function of α for $0\leqq\alpha\leqq 1$, and that various other relationships of similar type hold. *R. E. Fullerton* (College Park, Md.).

Slobodyanskiĭ, M. G. Estimate for eigenvalues in non-self-adjoint boundary problems. Prikl. Mat. Meh. **21** (1957), 330–338. (Russian)

The method of obtaining the indicated estimate is based on introducing certain "approximate" operators in a manner similar to that of the article in Prikl. Mat. Meh. **19** (1955), 295–314 [MR **17**, 286].

Author's summary.

Aronszajn, N.; and Smith, K. T. Characterization of positive reproducing kernels. Applications to Green's functions. Amer. J. Math. **79** (1957), 611–622.

Let $\mathfrak{F}$ be a proper functional Hilbert space, i.e., a Hilbert space whose elements u are functions on a set $\mathfrak{E}$ such that, for every x, $u(x)$ is a linear continuous functional on $\mathfrak{F}$. To such a space there corresponds a unique kernel $K(x, y)$ on $\mathfrak{E}\times\mathfrak{E}$ such that, for each y, $K(x, y)$ belongs to $\mathfrak{F}$ and possesses the reproducing property: if $u\in\mathfrak{F}$, then $u(y)=[u, K_y]$ [cf. N. Aronszajn, Trans. Amer. Math. Soc. **68** (1950), 337–404; MR **14**, 479]. The basic theorem of the note is that, in order that the reproducing kernel $K(x, y)$ for such an $\mathfrak{F}$ be real and non-negative on $\mathfrak{E}\times\mathfrak{E}$, it is necessary and sufficient that (a) for every u in $\mathfrak{F}$, $\bar{u}$ also be in $\mathfrak{F}$ and $\|u\|=\|\bar{u}\|$; (b) for u in $\mathfrak{F}$, there exists a $\tilde{u}$ in $\mathfrak{F}$ such that $\tilde{u}(x)\geqq|u(x)|$ for all x, and $\|\tilde{u}\|\leqq\|u\|$. Pseudo-reproducing kernels are introduced to take care of the "almost everywhere" situation as found for instance in L^2. If μ is a σ-finite complete measure on $\mathfrak{E}$, and $\mathfrak{E}'$ consists of the measurable subsets of $\mathfrak{E}$ of finite μ-measure, $\int_{x'} u\,d\mu$ (x' in $\mathfrak{E}'$) sets up a related Hilbert space of functions on $\mathfrak{E}'$, a reproducing kernel $K'(x', y')$ on $\mathfrak{E}'\times\mathfrak{E}'$, and a pseudo-reproducing kernel on $\mathfrak{E}\times\mathfrak{E}$ such that $K'(x', y')=\int_{x'}\int_{y'} K(x, y)d\mu(x)d\mu(y)$. The fundamental theorem is adapted to this extension by the addition of "almost everywhere". Application is made to Green's functions, $G(x, y)$ being a Green's function of a self-adjoint system $(A; B_i)$, consisting of an elliptic differential operator A and boundary conditions B_i relative to a domain D, if the closure of A has a bounded inverse which is expressible as an integral operator $\int Gf$. It is shown that if A is an operator of order 2, the following are equivalent: (1) the system $(A; B)$ is positive definite; (2) the Green's function for the system $(A; B)$ exists and is a pseudo-reproducing kernel; (3) the Green's function for the system $(A; B)$ exists and is non-negative. The only cases of higher order which are considered are of the form $(A^2; B_0, B_2)$ or

$(A^2; B_1, B_3)$, where A is as above, $B_0u=u$, $B_2u=-c(p(s))^{-2}\partial u/\partial v+(p(s))^{-1}Au$, $B_1u=\partial u/\partial v+bu$, $B_3u=d(p(s))^{-2}u+(p(s))^{-1}B_1Au$. Then, if $(A; B_0)$ $[(A; B_1)]$ and $(A^2; B_0, B_2)$ $[(A^2; B_1, B_3)]$ are both positive definite, and $c\geqq 0$ $[d\geqq 0]$, the Green's function for $(A^2; B_0, B_2)$ $[(A^2; B_1, B_3)]$ exists and is non-negative.

T. H. Hildebrandt (Ann Arbor, Mich.).

Pachale, Helmut. Zur Theorie der linearen Gleichungen im Hilbert-Raum. Math. Nachr. **15** (1956), 201–208.

Let H be a complete inner-product space and let K be a bounded linear operator defined on H. In terms of two carefully constructed, well-ordered, orthonormal bases in H, the author is able to state conditions on an f in the manifold orthogonal to $K^{*-1}(0)$ which are necessary and sufficient that the linear equation $Ky=f$ have a solution in H.

M. M. Day (Urbana, Ill.).

★ **Вайнберг, М. М. [Vaĭnberg, M. M.] Вариационные методы исследования нелинейных операторов. [Variational methods for investigation of non-linear operators.]** Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow, 1956. 344 pp. 10.65 rubles.

Techniques for finding solutions of operator equations by exhibiting them as extremal points of functionals and making use of the theory of differentiation of operators are discussed. A historical introduction is followed by two chapters on continuity and differentiability of functionals; these expand an article of the author [Uspehi Mat. Nauk (N.S.) **7** (1952), no. **4**(50), 55–102; MR **14**, 384]. They include a theory of a Riemann-Stieltjes integral with operator integrand. Operators which are gradients of functionals are discussed, and sufficient conditions, one on the differential of the operator and the other on its integral, for an operator to be of this type are given. Extrema are discussed in the 3rd, conditional extrema in the 4th chapter; applications to eigenvalue problems and an account of the category method of Lyusternik and Šnirelman are given. This is followed by an account of implicit functions and of branch points. The 7th and 8th chapters deal with operators of special types: the operator h, where $h(v)=g(v(x))$, named for Nemyckiĭ, and the integral operators of Lichtenstein and Hammerstein.

J. L. B. Cooper (Cardiff).

Bremermann, H. J. Holomorphic functionals and complex convexity in Banach spaces. Pacific J. Math. **7** (1957), 811–831.

Die Arbeit dehnt grundlegende Sätze der Funktionentheorie mehrer Veränderlichen auf komplexwertige G-holomorphe Funktionale in Banach-Räumen B_C aus. Nach Definition eines Gebietes D über einem B_C beweist der Verfasser die eindeutige Fortsetzbarkeit eines G-holomorphen Funktionals in Gebieten über einem B_C. Als Hilfsmittel dient weiter das reelle Funktional $d_D{}^N(z)$, das die Entfernung des Punktes z vom Rand des Gebietes D gemessen in der Norm N angibt, sowie $d_{aD}{}^N(z)$, das den Radius des größten in D liegenden Kreises um den Punkt z auf der Ebene $\{z': z'=z+\lambda a\}$ angibt, $d_D{}^N(z)$ ist stetig, $d_{aD}{}^N(z)$ ist halbstetig nach unten. Ist $-\log d_{aD}{}^N(z)$ für jedes a plurisubharmonisch, so heißt D pseudokonvex. Jedes Holomorphiegebiet ist pseudokonvex. D ist dann und nur dann pseudokonvexs falls $-\log d_D{}^N(z)$ für eine beliebige Norm N, die eine äquivalente Topologie erzeugt, plurisubharmonisch ist. Für jedes pseudokonvexe Gebiet gilt der sogenannte Kontinuitätssatz. Ferner erhält man: D ist dann und nur dann konvex, falls $-\log d_D{}^N(z)$ ein konvexes Funktional ist, oder falls $-\log d_{aD}{}^N(z)$ für jedes a ein konvexes Funktional ist, oder falls ein gewisser Kontinuitätssatz gilt. Die sogenannten Tubengebiete sind genau dann pseudokonvex, wenn sie konvex sind.

H. Griesel.

See also: Partial Order, Lattices: Kaplansky. **Integral Transforms:** Herz; Devinatz. **Ordinary Differential Equations:** Krasnosel'skiĭ, Kreĭn, and Sobolevskiĭ. **Partial Differential Equations:** Sobolev. **Integral and Integrodifferential Equations:** Čerskiĭ. **Topological Vector Spaces:** Bessaga and Pałczyński; Wiweger; Borisovič. **General Topology:** Borsuk. **Probability:** Getoor; Kampé de Fériet.

TOPOLOGY

General Topology

Borsuk, K. On the k-independent subsets of the Euclidean space and of the Hilbert space. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 351–356, XXIX. (Russian summary)

A subset A of Euclidean E_n or of Hilbert space is k-independent, where k is an integer, provided every subset of A consisting of $\leq k+2$ points is linearly independent. If A is k-independent for every k then it is said to be ω-independent. In this paper the author proves first that in a separable metric space X the set of homeomorphisms of X into ω-independent subsets of the Hilbert cube Q_ω is dense in the space $Q_\omega{}^X$ of all mappings of X into Q_ω. Interesting examples of k-independent subsets of Euclidean spaces are given, among them being a 2-independent simple closed curve in E_4. In this connection it is deduced that every plane set is homeomorphic with a k-independent subset of E_{2k+2}. Finally, it is shown that if A is a k-independent compact subset of E_n then (a) A is homeomorphic with a subset of E_{n-k+2}, and (b) for every open set U in E_n containing at least k points of A, the set $A-U$ is homeomorphic with a subset of E_{n-k}.

G. T. Whyburn.

de Albuquerque, Luis Mendonça. On two groups of set functions of the first class and axiomatics of more general abstract spaces. Las Ciencias **17** (1952), 639–655. (Portuguese)

Let X' denote the derived set of the set X in a Fréchet topological space. The author then studies set functions which are divided into two sets ("groups"): (1) $a(X)=X\cup X'$, $g(X)=X\cup(1-X')$, $k(X)=(1-X)\cup(1-X)'$, $m(X)=(1-X)\cup(1-(1-X)')$, $b(X)=X\cap(1-X)'$, $o(X)=(1-X)\cap X'$, $i(X)=X\cap(1-(1-X)')$, and $e(X)=(1-X)\cap(1-X')$; (2) $v(X)=X\cup(1-X)'$, $s(X)=X'\cup(1-X)$, $u(X)=(1-X)\cup(1-X')$, $t(X)=X\cup(1-(1-X)')$, $c(X)=X\cap X'$, $h(X)=X\cap(1-X')$, $n(X)=(1-X)\cap(1-X)'$, and $p(X)=(1-X)\cap(1-(1-X)')$. The axiomatics of very general topological spaces and properties of sets in such spaces are studied with the aid of some of the above functions. Some of the results are as follows. $g(X)$ has the properties: $g(0)=1$, $X\subset g(X)$, and X' consists of all $x\in 1$ such that $X-(x)\neq 0$ and $x\notin g(X-(x))$. If g is any set function defined over the set of all subsets of 1 such that $g(0)=1$, $X\subset g(X)$, and if X' is defined as in the preceding sentence, then $[1, ']$ is a Fréchet topological space such that $g(X)=X\cup(1-X')$. $(X\cup Y)'\subset X'\cup Y'$ if and only if $h(X)\cap h(Y)\subset X\cap Y\cap h(X\cup Y)$, for all $X, Y\subset 1$.

H. Tong.

de Albuquerque, Luís M. Correction to the article on isolated sets. Rev. Fac. Ci. Univ. Coimbra **23** (1954), 56–59. (Portuguese)

This note corrects an error in the paper reviewed above. The "proof" that "$h(X)\cap h(Y)\subset X\cap Y\cap h(X\cup Y)$" is equivalent to "$X\subset Y$ implies $h(X)\subset h(Y)$" is incorrect. Counterexamples can be found without difficulty. Two propositions involving this error are restated. Other comments are also included. *H. Tong* (New York, N.Y.).

Mrówka, S. A necessary and sufficient condition for $\mathfrak{m}$-almost-metrisability. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 627–629, LIII. (Russian summary)

The problem is to characterize those completely regular spaces X for which there is a compatible uniformity with base of cardinal $\mathfrak{m}$. This problem, for normal spaces X, has been solved by the author [same Bull. **5** (1957), 129–132, XII; MR **19**, 299] and independently by Chan-Chen-Hon in a paper to appear in Dokl. Akad. Nauk SSSR. Call a family $\mathscr{A}$ of subsets of X functionally locally finite [functionally discrete] if there is a system $\{f_U, U\in\mathscr{A}\}$ of continuous functions on X to $[0, 1]$ such that $f_U(x)=1$ for $x\in U$ and the family of all sets $\{x: f_U(x)>0\}$ is locally finite [discrete]. Then there is a compatible uniformity with base of power $\mathfrak{m}$ if and only if there is a base for the topology of X which is the union of $\mathfrak{m}$ functionally locally finite families, and this is the case if and only if the topology has a base which is the union of $\mathfrak{m}$ functionally discrete families. *J. L. Kelley.*

Rudin, Mary Ellen; and Klee, V. L., Jr. A note on certain function spaces. Arch. Math. **7** (1957), 469–470.

If X, Y are topological spaces, let $C(X, Y)$ denote the space of all continuous mappings of X into Y, under the topology of pointwise convergence. It is proved that if X, Y are separable metric, then $C(X, Y)$ is perfectly normal, and every subset of $C(X, Y)$ has the Lindelöf property. Corollary: Under its weak topology, every separable normed space has the Lindelöf property.

M. Katětov (Prague).

Whyburn, Gordon T. Relative quasi-compactness of mappings. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 974–978.

A mapping $f(X_0)=Y_0$ between topological spaces X_0 and Y_0 is quasi-compact provided that the image of each closed inverse set in X_0 is a closed set in Y_0. In this paper the quasi-compactness property is relativised as follows. Given topological spaces X and Y and subsets $X_0\subset X$ and $Y_0\subset Y$, a quasi-compact mapping $f(X_0)=Y_0$ is said to be quasi-compact relative to (X, Y) provided that, for each inverse set K in X_0 which is closed in X, $f(K)$ is closed in Y, and, likewise, for each set H in Y_0 which is closed in Y, $f^{-1}(H)$ is closed in X. Results about the invariance, transitivity and extension of relative quasi-compactness of mappings are obtained. Applications of the results to the special case in which the mapping is a homeomorphism are indicated. *W. T. Puckett* (Los Angeles, Calif.).

Tumarkin, L. A. On a universal metric space for compacta. Vestnik Moskov. Univ. **11** (1956), no. 2, 15–19. (Russian)

A negative answer is given to the question whether it is possible to construct a compact metric space containing an isometric image of every compact metric space with diameter less than a given positive constant.

From the introduction.

Skornyakov, L. A. Systems of curves on a surface. Trudy Moskov. Mat. Obšč. **6** (1957), 135–164. (Russian)

Contains a detailed and non-trivial proof (with argument of the R. L. Moore type) for the theorem announced without proof in Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 25–26 [MR **16**, 388]. *H. Busemann.*

Harrold, O. G., Jr. Locally tame curves and surfaces in three-dimensional manifolds. Bull. Amer. Math. Soc. **63** (1957), 293–305.

This paper describes many of the results obtained in the search for suitable analogues for higher dimensional Euclidean spaces of the Schoenflies extension theorem; i.e., results about the notion of "tameness". A detailed discussion is given of results of the author and his school involving implications of various types of local tameness. Several paths for future work in this connection are suggested. Regarding the problem discussed in Sec. 4, it may be noted that R. H. Bing has recently announced [Not. Amer. Math. Soc. **6** (1958), 365] the theorem that, if the complement of a 2-manifold M imbedded in Euclidean 3-space is locally simply connected at each point of M, then M is tamely imbedded. *E. Dyer.*

Sternberg, Shlomo. On Poincaré's last geometrical theorem. Proc. Amer. Math. Soc. **8** (1957), 787–789.

On the basis of a remark by Wintner [C. R. Acad. Sci. Paris **243** (1956), 835–836; MR **18**, 326] Poincaré's theorem may be stated as follows. Let $(R', \theta')=T(R, \theta)$ be a homeomorphism of period 2π in θ of the strip $A\leq R\leq B$ of the (R, θ) plane into this plane. The lines $R=A$ and $R=B$ are supposed to be invariant under T; assume, moreover, $\theta'>0$ for $R=A$ and $\theta'<0$ for $R=B$. For any set U let $|U|$ be its measure. Finally let α be the minimum of $|T(U)|/|U|$ if U varies over all open sets of the strip, and δ the minimum of the distances from a point of the strip to its image under T. The theorem then states that $\lim_{\alpha\to 1}\delta=0$. In the present paper it is shown that $\delta\leq(1-\alpha)(B-A)$ and that this inequality cannot be improved. *E. H. Rothe.*

Schwartzman, Sol. Asymptotic cycles. Ann. of Math. (2) **66** (1957), 270–284.

Let X be a compact space acted on by a continuous flow: $x\to xt$. Let $C(X)$ be the set of continuous complex valued functions f on X with $|f(x)|=1$, and let $R(X)$ consist of those f in $C(X)$ that admit a single-valued continuous logarithm. For any p in X let $\Delta_{(p, p0)}\arg f$ denote the change in the angle of $f(x)$ along the orbit from p to $p0$. For any quasi-regular point p, the $\lim_{t\to\infty}\Delta_{(p, p0)}\arg f$ exists and defines a group homomorphism of $C(X)/R(X)$ into the reals. Since $C(X)/R(X)$ can be identified with the set of elements of the first cohomology group with integral periods, this homomorphism can be extended to a linear functional A_p on the first Čech cohomology group with real coefficients. If μ is any normalized invariant measure then $A_\mu(f)=\int A_p(f)d\mu(p)$ is likewise such a functional. In case the first Betti number of X is finite, A_μ can be identified with an element of the first homology group, and this is called the asymptotic cycle corresponding to μ. Any continuous function can be uniformly approximated by a function differentiable with respect to the flow, and for such an f, $A_\mu(f)=(2\pi i)^{-1}\int(f'(p)/f(p))d\mu(p)$, which can sometimes be computed without detailed knowledge of the flow. In case f is an eigenfunction of the flow, $A_\mu(f)$ is equal to the corresponding eigenvalue. The flow is called spectrally determinate if A_μ is independent of μ. Conditions for this are found and it is shown that any

recurrent (i.e., almost periodic) flow in an arcwise connected continuum X is spectrally determinate. A flow admits a cross section if and only if $A_\mu(f)>0$ for every μ. A Hamiltonian flow on the torus is spectrally determinate and the coefficients of its asymptotic cycle turn out to be simply the winding numbers of Poincaré. In general, the asymptotic cycles describe how an average orbit winds around the space X. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Grabar', M. I. On a sufficient test for an isomorphism of dynamical systems. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 431–433. (Russian)

The author continues his studies [same Dokl. (N.S.) **109** (1956), 250–252; MR **18**, 662] of dynamical systems with identical trajectories. Using the theory of one-parameter groups of unitary operators he finds an elementary condition that two such systems be isomorphic.

G. Hufford (Stanford, Calif.).

See also: Functions of Complex Variables: Gahov and Krikunov. **Ordinary Differential Equations:** Bielecki. **Topological Vector Spaces:** Borisovič. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Huhunaišvili.

Algebraic Topology

★**Wallace, Andrew H. An introduction to algebraic topology.** Pergamon Press, New York-London-Paris, 1957. vii+198 pp. $6.50.

This is an elementary textbook on algebraic topology. It is intended for use in a first year course in the subject and assumes very little in the way of preparation, not even point-set topology being required as a prerequisite.

The first three chapters are devoted to a brief sketch of the point set topology needed later. The fourth chapter discusses the fundamental group of a topological space, and the next four chapters develop singular homology theory and show that it satisfies the Eilenberg-Steenrod axioms for homology theory (without mentioning the axioms explicitly). The last chapter shows how the homology groups of a finite simplicial complex can be calculated by using the properties of the singular homology groups established earlier.

The book is quite readable and there is a good deal of discussion to assist the reader to understand why a definition is made in a certain way or what the importance is of a particular theorem. Problems are spread throughout the book, and these form an integral part of the text, hints being provided for the solution of some of the more difficult ones.

The presentation of homology theory is such that the student cannot compute any non-trivial examples until the end of the book. This may make it harder for him to build up a geometric intuition for the subject. However, the singular homology groups are easily understood as an introduction to homology theory, so that it is a matter of taste whether the arrangement of the text or some other arrangement of material is preferred.

A more serious objection to the book arises from the fact that with little more effort the proof in the last chapter could have been modified so as to remain valid for any homology theory satisfying the Eilenberg-Steenrod axioms. Therefore, a better procedure might have been to introduce the axioms earlier and prove this more general result in the last chapter. On the other hand, a direct proof that the singular homology groups are isomorphic to the combinatorial groups for a finite simplicial complex could have been given in less space using the technique of simplicial approximation. This would have left room for the inclusion of more material. Either of these approaches seems preferable to the reviewer to the one actually adopted.

There are several topics not discussed which should, perhaps, be included in a first course. For example, there is no mention of the Euler-Poincaré characteristic. Neither is there any treatment of homology groups with coefficient group other than the integers, nor is there any cohomology theory.

These and other similar omissions make the book incomplete. It is, nonetheless, useful, and the interested student can learn from it. He should, however, be aware of the fact that he is getting only a brief introduction to the subject. *E. H. Spanier* (Chicago, Ill.).

★**Deheuvels René. Classes caractéristiques d'une application continue.** Colloque de topologie algébrique, Louvain, 1956, pp. 121–133. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

Let $0\to A\to B\to C\to 0$ be an exact sequence of sheaves of abelian groups on a topological space X. The sequence is said to be locally trivial if there exists an open covering of X such that the sequence splits over each element of the covering. If the sequence is locally trivial, then for any sheaf F there is an exact sequence of sheaves

$$0\to\mathrm{Hom}(F;A)\to\mathrm{Hom}(F;B)\to\mathrm{Hom}(F;C)\to 0.$$

Letting X be paracompact and applying the above to the case where $F=C$, there is the exact cohomology sequence

$$\cdots\to H^0(X;\mathrm{Hom}(C;B))\to H^0(X;\mathrm{Hom}(C;C))\xrightarrow{\delta^*}H^1(X;\mathrm{Hom}(C;A))\to\cdots.$$

The image $\gamma=\delta^*\Phi$ of the identity homomorphism Φ of C is called the characteristic class of the sequence of sheaves. Any element of $H^1(X;\mathrm{Hom}(C;A))$ is the characteristic class of a locally trivial extension of C by A, so that, in this way, $H^1(X;\mathrm{Hom}(C;A))$ is identified with the group of equivalence classes of locally trivial extensions of the sheaf C by the sheaf A.

The main result asserts that, for the exact cohomology sequence of X

$$\cdots\to H^n(X;B)\to H^n(X;C)\xrightarrow{\delta^*}H^{n+1}(X;A)\to H^{n+1}(X;B)\to\cdots$$

corresponding to the locally trivial sequence of sheaves $0\to A\to B\to C\to 0$, the homomorphism δ^* is the cup product by the characteristic class γ in the pairing

$$H^n(X;C)\otimes H^1(X;\mathrm{Hom}(C;A))\to H^{n+1}(X;A).$$

This is applied to the spectral sequence of a continuous map $\xi:Y\to X$. With suitable assumptions on ξ there is defined a sequence of characteristic classes $\gamma^p(\xi)$ which are 1-dimensional cohomology classes of X with coefficients in a sequence of sheaves determined by ξ. In certain cases these characteristic classes allow simple expressions of the differentials of the spectral sequence. For example, if ξ is a fibering with connected fiber and all these characteristic classes vanish, then all the differentials are zero. This gives a proof of the Leray-Hirsch theorem. In the case where the fiber is a sphere S^n the characteristic class $\gamma^n(\xi)$ can be identified with the customary characteristic class of the sphere fibering.

There is a final section containing a proof that for a paracompact X the Čech and Alexander cohomology groups of X with coefficients in the same sheaf are isomorphic. *E. H. Spanier* (Chicago, Ill.).

Adams, J. F. Une relation entre groupes d'homotopie et groupes de cohomologie. C. R. Acad. Sci. Paris **245** (1957), 24–25.

The author announces the theorem: If both $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ have elements with Hopf invariant 1, then $n \leq 4$. This is based on the following general construction. Let X be a space with finitely generated homology, p a fixed prime, $A=A_p$ the graded Steenrod algebra, and $H^*=H^*(X, Z_p)$ the singular cohomology of X. Regarding Z_p as a trivial A-module, $\mathrm{Ext}_A^{s,t}(H^*, Z_p)$ is bigraded, with degree t that of H^t and s that of $\mathrm{Ext}_A^s(L,M)$. Let S^nX be the n-fold suspension of X and $\bar{\pi}_m(X)=\varinjlim_n \pi_{m+n}(S^nX)$ the "stable" homotopy group. The author asserts the existence of a spectral sequence with $E_2^{s,t}=\mathrm{Ext}_A^{s,t}(H^*, Z_p)$ and of a suitable filtration $\{B^{s,m+s}\}$ of $B^{0,m}=\bar{\pi}_m(X)$ such that $E_\infty^{s,t}=B^{s,t}/B^{s+1,t+1}$. In case X is the zero-sphere, $E_2^{s,t}=\mathrm{Ext}_A^{s,t}(Z_p, Z_p)=H^{s,t}(A)$ and there are associative products $E_r^{s,t}\cdot E_r^{s',t'}\to E_r^{s+s',t+t'}$ $(r\leq\infty)$. If, in addition, $p=2$, then, setting $H^s(A)=\sum_t H^{s,t}(A)$ and denoting the Z_p-base of $H^1(A)$ by $\{h_m\}$, each h_m has t-degree 2^m, and further $h_m\in E_\infty^{1,n}$, $n=2^m$, if and only if there is an element of $\pi_{2n-1}(S^n)$ with Hopf invariant 1. The author then states the linear relations between the products $h_i\cdot h_j\in H^2(A)$ and between the products $h_i\cdot h_j\cdot h_k\in H^3(A)$, from which the announced theorem follows at once. *J. Dugundji.*

★ **Hilton, P. J. Generalizations of the Hopf invariant.** Colloque de topologie algébrique, Louvain, 1956, pp. 9–27. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

In the course of twenty-five years the original idea of the Hopf invariant has inspired various generalizations. In his survey the author compares these, beginning with an up-to-date treatment of the classical case. He discusses various relations, some of them new, and points out a number of problems. {Reviewer's note: some of these have since been solved by M. G. Barratt (unpublished).} *I. M. James* (Cambridge, England).

★ **Eckmann, Beno. Homotopie et dualité.** Colloque de topologie algébrique, Louvain, 1956, pp. 41–53. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

This is a survey without proofs of the author's work with P. J. Hilton on duality. A definition of homotopy for homomorphism of modules, which is suggested by analogy with topology, leads to the introduction of algebraic concepts of homotopy type, homotopy groups, suspension, and fibration. The duality in module theory between projective and injective modules provides a duality in this algebraic homotopy theory which, in turn, can be interpreted as a duality in topological homotopy theory. The resulting duality in topology (under which homotopy groups are dual to cohomology groups) is not precise, but more of a metamathematical principle that corresponding to a theorem there is a dual theorem (each of these dual theorems being proved separately).

The first part is devoted to the algebraic duality. If A and B are left unitary modules over a ring Λ with 1, then (A, B) denotes the module of homomorphisms over Λ of A into B. $f, g\in(A, B)$ are said to be (1) i-homotopic if $f-g$ can be factored by any monomorphism $A\to A'$, and are said to be (2) p-homotopic if $f-g$ can be factored by any epimorphism $B'\to B$. Each of these is an equivalence relation in (A, B), and the corresponding homotopy classes are the elements of the group $\bar{\pi}(A, B)$ for (1) and $\underline{\pi}(A, B)$ for (2). Corresponding to each of these concepts of homotopy is a concept of homotopy type of modules called i-equivalence and p-equivalence, respectively. The suspension SA of a module A is defined to be the quotient $\bar{A}/A$, where $\bar{A}$ is an injective module containing A. The i-type of SA depends only on that of A and, by iteration, the same is true of the successive suspensions S^nA. The group $\bar{\pi}(S^nA, B)$ is called the nth injective homotopy group of A in B and is denoted by $\bar{\pi}_n(A, B)$. By a dual construction ΩB is defined and the nth projective homotopy group of A in B is defined by $\underline{\pi}_n(A, B)=\underline{\pi}(A, \Omega^nB)$. These concepts are then extended to the relative case and suitable exact sequences are formed. The concepts of i-fibration and p-fibration are introduced, and the corresponding exact sequences involve only the appropriate absolute groups.

The second part shows how these concepts give dual results in topology. In this case the two kinds of homotopy both reduce to ordinary homotopy of mappings and the homotopy groups $\pi(S^nA, B)$ and $\pi(A, \Omega^nB)$ are isomorphic. Corresponding to the exact sequences of the algebraic case are exact sequences of homotopy classes. These include the usual homotopy and cohomotopy sequences of a pair. Using fibrations it is shown that homotopy and cohomology are dual. *E. H. Spanier.*

Berstein, Israël. Suites spectrales pour les espaces des applications continues. C. R. Acad. Sci. Paris **245** (1957), 880–882.

The main theorem of this note asserts the existence of a spectral sequence which relates the homotopy groups of the function space Y^X and the cohomology groups of X with coefficients in the homotopy groups of Y. This spectral sequence is very similar to one described in an earlier paper by H. Federer [Trans. Amer. Math. Soc. **82** (1956), 340–361; MR **18**, 59]. Apparently the author's method of deriving this spectral sequence is essentially the same as the method of Federer. *W. S. Massey.*

★ **Dedecker, Paul. Groupoïdes de cohomologie à coefficients non abéliens et espaces fibrés.** Colloque de topologie algébrique, Louvain, 1956, pp. 135–149. Georges Thone, Liège; Masson & Cie, Paris, 1957. 375 fr. belges; 3000 fr. français.

It is known [Dedecker, Bull. Soc. Math. Belg. **6** (1953), 97–125; MR **17**, 80] that the set of classes of principal fibre bundles with base B and structural group G are in 1-1 correspondence with the elements of $H^1(B; \mathfrak{G})$, where $\mathfrak{G}$ is the sheaf of jets of local sections of B in G. When $\mathfrak{G}$ is non-abelian, $H^1(B; \mathfrak{G})$ is only a set with a distinguished element (corresponding to the trivial bundle $B\times G$). Corresponding to an exact sequence of sheaves $e\to\mathfrak{N}\to\mathfrak{G}\to\mathfrak{H}\to e$ there is an exact sequence $e\to H^0(B; \mathfrak{N})\to H^0(B; \mathfrak{G})\to H^0(B; \mathfrak{H})\to H^1(B; \mathfrak{N})\to H^1(B; \mathfrak{G})\to H^1(B; \mathfrak{H})$. In this paper, the author naturally embeds $H^1(B; \mathfrak{G})$ in a groupoid $\mathfrak{H}^1(B; \mathfrak{G})$ (a groupoid is a set with a sometimes defined operation which is associative, has cancellation, and which has left and right identities and inverses). $\mathfrak{H}^1(B; \mathfrak{G})$ operates on $H^1(B; \mathfrak{G})$ on the left in such a way that $H^1(B; \mathfrak{G})$ is the set of elements in $\mathfrak{H}^1(B; \mathfrak{G})$

for which there are right identities. Also, the author shows that the exact sequence above can be replaced by an exact sequence of groupoids. *F. P. Peterson.*

Pavel, Monica. Petites déformations des espaces fibrés. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 2 (1953), no. 3, 34–36. (Romanian. Russian and French summaries)

A metric space E is stable [Alexandroff and Hopf, Topologie, Bd. I, Springer, Berlin 1935, p. 524] if, for some $\varepsilon>0$, any deformation of E on itself with trajectories of diameter less than ε has all of E as final image. The author proves that if X is a stable compact metric fibre space over B then B is also stable. *J. M. Danskin.*

★ **Haefliger, André. Sur les feuilletages des variétés de dimension n par des feuilles fermées de dimension $n-1$.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 8 pp.

The author considers manifolds not necessarily satisfying the Hausdorff separation axiom. Such manifolds often occur as quotient spaces of a foliation of a manifold; several such cases are studied. A foliation is orientable if, roughly speaking, the transversal spaces to the leaves can be oriented coherently. If V is a separated (Hausdorff axiom) manifold with countable basis, then every closed leaf L of a foliation is shown to be proper (the inclusion LCV is a homeomorphism). Main results: Let V be a separated manifold of dimension n and finite first Betti number p_1 mod 2, foliated with closed leaves of dimension $n-1$; if the foliation is orientable, the quotient space is an orientable 1-dimensional manifold; in the non-orientable case at most p_1 endpoints may appear; if $p_1=0$, the leaves are level surfaces of a continuous real function. [There is overlap with the work of G. Reeb, Colloque de topologie de Strasbourg, 1954–1955, Inst. Math., Univ. de Strasbourg; MR 19, 441.] *H. Samelson.*

Bourgin, D. G. Un indice dei punti uniti. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 21 (1956), 395–400.

[For Parts I-II see same Rend. (8) 19 (1955), 435–440; 20 (1956), 43–48; MR 17, 1120.] The author considers spaces which are limits of ANR's and obtains fixed-point theorems for such spaces. A compact subset X of a fixed parallelotope P is called an NR_3 if it is the intersection of a family $\{X_\alpha\}$ of ANR's, subsets of P, such that X_α is a retract of X_β whenever $X_\alpha CX_\beta$. Let X be an NR_3. Then X is homologically finite, so that every mapping $f: X\to X$ has a Lefschetz number $L(f)$. Assume now that there exists a family of mappings $t_\alpha: X_\alpha\to X$ which are compatible, in a certain homological sense, with the retractions $X_\beta\to X_\alpha$. Assume further that the t_α satisfy a uniformity condition which states roughly that t_α is near the identity when X_α is near X. Under these conditions the nonvanishing of $L(f)$ implies the existence of a fixed point. There are a number of interesting corollaries. *P. A. Smith* (New York, N.Y.).

Whitehead, J. H. C. On involutions of spheres. Ann. of Math. (2) 66 (1957), 27–29.

Let S_n be an n-sphere, f a mapping $S_n\to S_n$ of period p, F the fixed-point set of f. Assuming p to be a prime, it is known that F is a homology q-sphere over the integers mod p. For small values of n, F is actually a q-sphere. The author constructs an example showing that for sufficiently large n, F need not be a sphere. In the example p is 2, q is any integer ≥ 4, n is any integer $\geq 2q+6$, S_n is a polyhedral n-sphere, F is a manifold which is a subpolyhedron and has a non-trivial fundamental group (hence is not a sphere) and f is piecewise linear. (See also the example given by Floyd in the paper reviewed below.) *P. A. Smith* (New York, N.Y.).

Floyd, E. E. Fixed point sets of compact abelian Lie groups of transformations. Ann. of Math. (2) 66 (1957), 30–35.

Let S^n be an n-sphere, T a mapping $S^n\to S^n$ of period p^m, F the fixedpoint set of T. It is known that if p is a prime, F is a homology sphere over Z_p. The question whether F need be a homology sphere when the period of T is not a power of a prime is settled in the negative in the present note. The author constructs an example in which n is 41, T is of period 6 and F is the disjoint union of two connected sets, one of which has the homology of a 6-sphere (and therefore F is not a homology sphere over any non-trivial coefficient group). There is, moreover, a triangulation of S^n relative to which F is a subcomplex and T is simplicial.

Further results. Let G be a circle group operating on a compact n-manifold X. The fixedpoint set F is homology locally connected over R (reals modulo 1) in all dimensions. — Let G be a toral group acting on a compact manifold which is a homology n-sphere over R. The fixedpoint set, if it is not empty, is a homology r-sphere for some $r\leq n$. *P. A. Smith* (New York, N.Y.).

See also: General Topology: Schwartzman.

GEOMETRY

Geometries, Euclidean and Other

Serini, Rocco. Destra e sinistra nella fisica. Rend. Sem. Mat. Fis. Milano 26 (1954–55), 103–115 (1957).

The author reviews the notion of right and left in 3-space, dealing with vectors, skew-symmetric tensors, and the permutation symbol of Ricci. *J. L. Synge.*

Sedlák, František; und Kosmák, Ladislav. Studie der Fokale. I. Časopis Pěst. Mat. 82 (1957), 160–164. (Czech. Russian and German summaries)

Es wird ein elementarer Beweis der Aequivalenz zweier Definitionen der Fokale gegeben; nach der ersten wird die Fokale als der Ort der Brennpunkte der Kegelschnitte einer Schar definiert, nach der zweiten wird sie mittels gewisser Kreisbüschel erzeugt. Dies hängt mit der Aufgabe zusammen, den Ort aller Punkte der Ebene zu finden, von denen aus zwei Strecken unter gleichen Winkeln erscheinen. *Zusammenfassung des Autors.*

Alda, Václav. Les réseaux de coniques. Czechoslovak Math. J. 7(82) (1957), 48–56. (Russian summary)

Le théorème, que tout réseau de coniques est un réseau de courbes polaires d'une cubique, n'est valable qu'"en général". Dans cet article les conditions nécessaires et suffisantes pour la validité de ce théorème sont précisées et démontrées. *Résumé de l'auteur.*

Trost, E. Eine charakteristische Eigenschaft der Rotationsflächen zweiten Grades. Elem. Math. **12** (1957), 73–75.

Analytical demonstration of the following [cf. also Blaschke, Math. Nachr. **15** (1956), 258–264; MR **18**, 922]: A surface of revolution is such that the boundary of its shadow (on itself) under parallel illumination is planar if and only if it is generated by rotation of a conic section about one of its axes.

Trost, E. Über eine spezielle Hüllkurve. Elem. Math. **12** (1957), 97–100.

Remarks, with examples, on the fact that the chords which cut off a constant area from a fixed curve $\mathfrak{C}'$ generate an envelope $\mathfrak{C}$ whose points of tangency are the midpoints of the chords. In particular, if $\mathfrak{C}'$ is a conic section, $\mathfrak{C}$ is a homothetic conic section.

Drs, Ladislav. Über den Hauptsatz der zentralen Axonometrie. Časopis Pěst. Mat. **82** (1957), 165–174. (Czech. Russian and German summaries)

Der Absatz 1 dieser Arbeit enthält einen Beweis des Hauptsatzes zentraler Axonometrie. In diesem Satz sind die Bedingungen enthalten, welche im Absatz definierte δ-Konfiguration erfüllen muss, soll sie eine Zentralprojektion eines rechtwinkeligen Koordinatensystems sein (δ-Konfiguration vom Typus S).

Die Absätze 2, 3 enthalten Konstruktionen der δ-Konfiguration vom Typus S und im Absatz 4 löst man einige Aufgaben über diese δ-Konfigurationen und über verschiedene Arten von Perspektiven.

Zusammenfassung des Autors.

Havel, Václav. Die Fundamentalsätze der Zentralaxonometrie. Časopis Pěst. Mat. **82** (1957), 175–181. (Czech. Russian and German summaries)

In dieser Abhandlung werden die ebenen Desarguesschen Konfigurationen, sowie die Koordinatenkonfigurationen in dem ergänzten euklidischen Raum untersucht (die erwähnte Koordinatenkonfiguration besteht aus dem Nullpunkt, aus den Einheitspunkten auf den senkrechten Koordinatenachsen und aus den uneigentlichen Punkten der Koordinatenachsen). Weiter wird die sg. d-Konfiguration vom Typus T definiert, die als Projektion einer Koordinatenkonfiguration betrachtet werden kann.

Zunächst wird der erste Fundamentalsatz der Zentralaxonometrie (der sg. Satz von Kruppa-Četveruchin) verallgemeinert, der die notwendige und hinreichende Bedingungs dafür gibt, dass der gegebenen d-Konfiguration der Typus T zugehört. Die Formulation des Satzes erfordert die Einführung spezieller Begriffe und darum wird hier nicht eingeführt. Weiter ist der zweite Fundamentalsatz der Zentralaxonometrie (der sg. Satz von Beskin-Stiefel) abgeleitet. Nach diesem Satze steht jede d-Konfiguration mit einer d-Konfiguration vom Typus T in der projektiven Beziehung. Zum Beweis wird die Methode von Stiefel gebraucht. Zum Schluss wird folgender Satz bewiesen: Jede d-Konfiguration in der Ebene ρ ist mit einer d-Konfiguration vom Typus T in der Ebene ρ in der Beziehung der perspektiven Affinität. Auf diese perspektive Affinität können spezielle Forderungen gelegt werden. *Zusammenfassung des Autors.*

Havel, Václav. Eine Bemerkung zum Staudtschen Satz in der Moufang-Ebene. Czechoslovak Math. J. **7(82)** (1957), 314–317. (Russian summary)

The author proves that a Staudt projectivity of a line p upon a line p', where p, p' lie in a Moufang plane of characteristic not 2, is a product of a projectivity of p upon p' and a mapping of p' upon itself induced by a certain type of mapping σ of a suitable (alternative division) coordinate ring A of the plane. Specifically, σ is a one-to-one mapping of A upon A which satisfies the identities (1) $(x+y)\sigma=x\sigma+y\sigma$, (2) $(x^2)\sigma=(x\sigma)^2$, and hence the identities (3) $(xy+yx)\sigma=(x\sigma)(y\sigma)+(y\sigma)(x\sigma)$, (4) $(xyx)\sigma=(x\sigma)(y\sigma)(x\sigma)$. Moreover, any such "Jordan-semi-automorphism" σ will serve. In an earlier paper [same J. **5(80)** (1955), 83–90; MR **17**, 522] the author outlined a proof that σ was an automorphism or anti-automorphism of A. Here he acknowledges an error pointed out in the review by L. A. Skornyakov [RŽMat. **1956**, no. 6833] and refers (without actually giving it) to a counter-example due to Skornyakov. *R. H. Bruck.*

Mandan, Sahib Ram. Möbius tetrads. Amer. Math. Monthly **64** (1957), 471–478.

The author exploits the fruitful idea of associating the Möbius pairs of tetrahedrons with a ruled quadric surface. Using both the synthetic and the analytic methods he derives in a rather simple way a number of interesting results. Most of them, however, have been anticipated by J. Steiner, J. Neuberg, and others [cf. W. Fr. Meyer, Math. Z. **2** (1918), 453–474, esp. footnotes on p. 453].

N. A. Court (Norman, Okla.).

Naumann, H.; und Reidemeister, K. Über Schliessungssätze der Rechtwinkelgeometrie. Abh. Math. Sem. Univ. Hamburg **21** (1957), 1–12.

A relation between lines g and h of an affine plane is called perpendicularity and written $g\perp h$ if it satisfies the orthogonality laws: (O1) If $g\perp h$, then $h\perp g$; (O2) if $g\perp h$ and $h\|k$, then $g\perp k$; (O3) through each point of a line g there is exactly one line h such that $g\perp h$.

This paper studies some incidence theorems in this "right angle" geometry. A) (Reidemeister Theorem) If P_i and P_i' ($i=0, 1, 2, 3$) are the vertices of two non-degenerate quadrilaterals and if five of the relations $P_iP_j\perp P_i'P_j'$ hold, then the sixth also holds. B) (Schutte Theorem) If P_i' and P_i'' ($i=0, 1, 2, 3$) are two sets of four points and if $P_0'P_i'=g_i'$ ($i=1, 2, 3$), $h_1'=P_2'P_3'$, $h_2'=P_3'P_1'$, $h_3'=P_1'P_2'$ and if g_i'', h_i'' are similarly defined for P_i'', then if five of the relations $g_i'\perp h_i''$, $g_i''\perp h_i'$ hold, the sixth also holds. Each of these theorems implies the theorem of Desargues. The condition for orthogonality in both cases is a condition $m_1cm_2=-1$ on the slopes m_1, m_2 of the two lines $y=m_jx+b_j$ ($j=1, 2$), where c is a fixed element and, when A) holds, the mapping $a\to\bar{a}$ is an automorphism of the coordinatizing division ring and $c\bar{\bar{a}}=ac$, while when B) holds it is an involutory anti-automorphism. A more restricted hypothesis is investigated: C) Let P_i, P_i' ($i=0, 1, 2, 3$) be the vertices of two non-degenerate quadrilaterals and suppose that $P_0P_1\|P_2P_3$. Then if five of the relations $P_iP_j\perp P_i'P_j'$ hold, the sixth also holds. C) is a special case of both A) and B). If the Theorem of Desargues is assumed as well as C), the orthogonality takes the form $m_1cm_2=-1$ where $\bar{c}\bar{\bar{a}}=ac$ and $a\to\bar{a}$ is a semi-automorphism, i.e., a mapping preserving addition and satisfying $\overline{\bar{a}\bar{b}}=\bar{a}\bar{b}\bar{a}$, $\bar{1}=1$. By a result of Hua [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 386–389; MR **10**, 675], the mapping must be an automorphism or an anti-automorphism and so either A) or B) must hold. But it is to be noted that the Theorem of Desargues was assumed along with C). *Marshall Hall, Jr.*

Fladt, Kuno. Die nichteuklidische Zyklographie und ebene Inversionsgeometrie. (Geometrie von Laguerre, Lie und Möbius.) I. Arch. Math. **7** (1956), 391–398.

Projection of the absolute A ($x_1^2+x_2^2-x_3^2=0$, $x_0=0$) ($x_0=0$: plane at infinity) from a point P over the plane $x_3=0$ gives an oriented circle C. The correspondence $C \pm P$ gives rise to the geometry of Laguerre and cyclography of Müller [Blaschke, Vorlesungen über Differentialgeometrie, Bd. III, Berlin, 1929, Kap. 4]. Taking as absolute A the quadric $x_0^2+k(x_1^2+x_2^2-x_3^2)=0$ ($k=\pm 1$), the author obtains analogous results for non-euclidean geometry. *L. A. Santaló* (Buenos Aires).

Fladt, Kuno. Die nichteuklidische Zyklographie und ebene Inversionsgeometrie. (Geometrie von Laguerre, Lie, und Möbius.) II. Arch. Math. **7** (1957), 399–405.

Following the paper reviewed above the author investigates the geometry of the groups of linear transformations $\rho x_i' = a_{ih} x_h$, $\rho x_4' = a_{44} x_4$ ($i, h=0, 1, 2, 3$) (non-euclidean Laguerre's geometry) and $\rho x_i' = a_{ih} x_h$, $\rho x_3' = a_{33} x_3$ ($i, h=0, 1, 2, 4$) (non-euclidean Möbius geometry) which leave invariant the equation $x_0^2+k(x_1^2+x_2^2-x_3^2)-x_4^2=0$, where x_i are the homogeneous pentacyclical coordinates of a cycle on the plane. *L. A. Santaló.*

See also: Differential Geometry: Blanuša. **Algebraic Geometry:** Svoboda.

Convex Domains, Integral Geometry

Steinhaus, H. On chords of convex curves. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 595–597, LI. (Russian summary)

It is shown that for every plane convex closed curve K and every three angles a, b, c with $a+b+c=\pi$, there exist three chords of K with a common midpoint, the angles between the consecutive halves being a, b, c, a, b, c. The question as to whether the theorem applies to every plane closed curve remains open. *L. Moser.*

Vincensini, Paul. Sur un invariant de partition de l'ensemble des corps convexes de l'espace euclidien à n-dimensions. C. R. Acad. Sci. Paris **245** (1957), 132–133.

In Euclidean n-space, let $\mathscr{E}$ be the set of all convex bodies whose bounding surfaces have everywhere continuously turning tangent hyperplanes and finite non-vanishing principal curvatures, and let $\mathscr{S}$ be the set of all members of $\mathscr{E}$ which are symmetric about the origin. For each $S \in \mathscr{S}$, let eS denote the set of all $C \in \mathscr{E}$ which have S as vector body (i.e., for which S is equal to the vector difference $C-C$). For each $S \in \mathscr{S}$, let fS denote the set of all functions of the form $g-h$, where g and h are support functions of members of eS. The author shows that for arbitrary $S_1, S_2 \in \mathscr{S}$, each function in fS_1 is a positive multiple of some function in fS_2. [For another result relating eS_1 and eS_2, used in the present proof, see the author's report in 'Colloque sur les questions de realité en géométrie, Liège, 1955', Thone, Liège, 1956, pp. 77–94; MR **17**, 1123.] *V. L. Klee, Jr.* (Seattle, Wash.).

De Santis, Richard. A generalization of Helly's theorem. Proc. Amer. Math. Soc. **8** (1957), 336–340.

The author obtains the following extension of the lemma used by Radon to establish Helly's theorem on the intersection of convex sets. If $F_1, \cdots, F_{k+2}$ are flats of deficiency k in a linear space E, there are complementary subsets I and J of $\{1, \cdots, k+2\}$ such that $(\text{conv } \bigcup_{i \in I} F_i) \cap (\text{conv } \bigcup_{j \in J} F_j)$ contains a flat of deficiency k. This leads to the following result. Suppose Φ is a finite family of convex sets in a linear space E, and each $k+1$ members of Φ contain a common flat of deficiency k. Then there is a flat of deficiency k in the intersection of all members of Φ. (The results of Radon and Helly are as above with $E=E^n$ and $k=n$.) *V. L. Klee, Jr.* (Seattle, Wash.).

Tsuji, Masatsugu. On a theorem of Fenchel on the total curvature of a closed curve. Comment. Math. Univ. St. Paul. **6** (1957), 29–32.

Let C be a closed space curve of class C^2 and $k(s)$ its curvature, s being the arc length measured from an arbitrary point of C. Fenchel [Math. Ann. **101** (1929), 238–252] proved that (*) $\oint k(s)ds \geqq 2\pi$ and the equality holds when and only when C is a plane convex curve. Several authors gave various proofs (H. Liebmann, B. Segre, K. Borsuk, H. Rutishauser–H. Samelson), and others gave generalizations of the theorem (I. Fary, J. Milnor and the reviewer). The author gives here an elementary proof of this theorem by means of a limiting process and the following lemma. Let Π_n be a closed space polygon of n sides and Θ_n be the sum of the exterior angles at its vertices; then $\Theta_n \geqq 2\pi$, and the equality sign holds when and only when C is a plane convex polygon.

The following theorem, which was used by Fenchel to prove (*), is also proved by virtue of the above lemma. Let C be a rectifiable closed space curve of length L on a unit sphere S, which is not a plane curve, such that the center O of S is an inner point of the smallest convex set which contains C; then $L > 2\pi$. *S. Sasaki.*

Huber, Alfred. Zur isoperimetrischen Ungleichung auf gekrümmten Flächen. Acta Math. **97** (1957), 95–101.

The author gives a simple proof of the following result, which he has previously [Ann. of Math. (2) **60** (1954), 237–247; **63** (1956), 572–587; MR **16**, 508; **18**, 296] presented in somewhat less general form. Let a conformal metric $ds=e^{u(z)}|dz|$ be defined on a Riemann surface R, and suppose that $u(z)$ can be represented locally as the difference of two subharmonic functions, $u(z)=u_1(z)-u_2(z)$. Let γ be a locally rectifiable Jordan curve on R, enclosing a simply connected domain ω. Let $\mu_2(e)$ be the measure associated with $u_2(z)$, and write $\mu_2(\omega)=\alpha$. Then the following inequality holds:

$$\left(\int_\gamma e^u |dz|\right)^2 \geqq 4\pi(1-\alpha) \iint_\omega e^{2u} dxdy,$$

where $z=x+iy$; the sign of equality is valid if and only if $\alpha<1$ and $u(z)=\log|\Phi'(z)(\Phi(z))^{-\alpha}|+c$, where $w=\Phi(z)$ is an arbitrary conformal map of ω on $|w|<1$ and c is an arbitrary real constant. *E. F. Beckenbach.*

See also: Approximations, Orthogonal Functions: Rémès. **Differential Geometry:** Alexits. **Programming, Resource Allocation, Games:** Goldman.

Differential Geometry

Alexits, G. Die Entwicklung des allgemeinen Raumbegriffes. Schr. Forschungsinst. Math. **1** (1957), 85–91.

This is principally a historical sketch. It discusses briefly the explicitly materialistic point of view of the discoverers of non-Euclidean geometry, Riemann's contri-

bution to geometry, the general concept of space by Fréchet, F. Riesz, Hausdorff, Urysohn's and Menger-Noebeling's imbedding theorems, Menger's concept of curvature of curves in metric spaces, Wald's metric characterization of Gaussian surfaces. It deals in greater detail with the author's and Egerváry's definition of higher curvatures and ends with some conjectures of these authors on the determination of a curve in E^n by these curvatures.

H. Busemann (Los Angeles, Calif.).

★ **Lagally, Max. Vorlesungen über Vektorrechnung. 5. Auflage, bearbeitet von Dr. Walter Franz.** Mathematik und ihre Anwendungen in Physik und Technik. Bd. 2. Akademische Verlagsgesellschaft, Leipzig, 1956. xx+462 pp. DM 22.00.

In the present 5th edition of this well-known book (the 4th edition appeared in 1928) only such changes are made as to bring the methods and subject-matter up to date. The most important changes were made in the chapter on hypercomplex numbers, which is now incorporated into the much larger Chapter 6 on n-dimensional vector spaces. Other additions are the sections on integration by parts, on quaternionic spherical harmonics, and on differential equations with vector variables. The chapter headings are: 1. Elementare Vektoralgebra. 2. Von skalaren Parametern abhängige Vektoren. 3. Theorie der Felder. 4. Dyaden. 5. Die wichtigsten Dyaden der Physik. 6. Vektorräume höherer Dimension und Cliffordsche Algebra. 7. Transformationstheorie. 8. Vektoren im Riemannschen Raum.

Pozzolo Ferraris, Giulia. Sopra due curve notevoli. Boll. Un. Mat. Ital. (3) **12** (1957), 46–49.

P étant un point quelconque d'une courbe gauche (γ), supposée rencontrée en deux points voisins P_1, P_2 par un plan (α) perpendiculaire à la normale principale en P et coupant cette normale au voisinage de P, la courbe lieu du milieu M du segment rectiligne P_1P_2 lorsque (α) varie est la courbe diamétrale de (γ) relative au point P. L'auteur détermine la direction de la tangente à la courbe diamétrale en P, et il rattache à cette détermination certaines propriétés des courbes planes établies par F. Tisserand.

P. Vincensini (Marseille).

Rozet, O. Sur les suites de Laplace et les grilles hyperboliques. Bull. Soc. Roy. Sci. Liège **26** (1957), 105–111.

The two systems of asymptotic tangents of a non-parabolic surface in an ordinary projective space are represented, on Klein's quadric in S_5, as two surfaces with a conjugate system, each of them being the Laplace transform of the other. The author proves that each surface belonging to the congruence of lines joining corresponding points on two consecutive Laplace transforms of the sequence has a double conjugate system of the second kind (a notion introduced by this reviewer). *E. Bompiani.*

Senkin, E. N. Concerning a property of bending of convex surfaces with a boundary. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 173–174. (Russian. English summary)

Let F_1, F_2 be two intrinsically isometric convex surfaces in E^3 of class C^2 and with positive Gauss curvature and with boundaries L_1, L_2. If F_1 is not congruent to F_2 then there are points a_1, a_1', b_1, b_1' on L_1 such that the corresponding points a_2, a_2', b_2, b_2' on L_2 (under the isometry) satisfy the inequalities $|a_1-a_1'|<|a_2-a_2'|$ and $|b_1-b_1'|>|b_2-b_2'|$, where $|x-y|$ is the distance of x and y in E^3. The hypothesis that the curvature be positive is essential. *H. Busemann* (Los Angeles, Calif.).

Ryžkov, V. V. Metric tangential bending of surfaces. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 763–765. (Russian)

The problem of the deformation of order k of two n-dimensional manifolds S_n and S_n' in euclidean space E_N $(N>n)$ with respect to a group of transformations G, which E. Cartan formulated [C. R. Congrès Internat. Math Strasbourg, 1920, Privat, Toulouse, 1921, pp. 397–406= Oeuvres complètes, partie III, v. 1, Gauthier-Villars, Paris, 1955, pp. 539–548; MR **17**, 697] has led the present author to his papers in Dokl. Akad. Nauk SSSR (N.S.) **75** (1950), 503–506 [MR **13**, 776] and Uspehi Mat. Nauk (N.S.) **5** (1950), no. 4(38), 134–135 [MR **12**, 358]. In the present paper, he considers the metrical group and the relations between point deformations and the corresponding tangential deformations. Every metrical tangential deformation of order $k>1$ is at the same time a point deformation of order $k+1$, and a point-tangential deformation of order one is at the same time a point deformation of order two. Surfaces of three-space which satisfy a metrical tangential deformation on a given one coincide but for a motion and a transformation of Peterson. The latter is defined as a transformation in which parallelism of corresponding tangent spaces is preserved. The analytical apparatus for the surfaces S_n with $x=x(u^i)$ and S_n' with $y=y(u^i)$ is developed from the formulas $Ax+b=y$, $Ad^sx=d^sy$ $(s=1, 2, \cdots, k-1)$, $Ad^kx=d^ky+\omega^\alpha y_\alpha$, $Ad^kx_i=d^ky_i+k\omega^\alpha y_{\alpha i}+\Omega^\alpha y_{\alpha}$ $(\alpha=1, 2, \cdots, n)$; here ω^α, Ω^α are differential forms of order k and $A=A(u^i)$ is the transformation matrix $\xi\to A\xi+b$. If $\omega^\alpha=0$ we have a point-tangential deformation. In the case $Adx=\lambda dy$, $Ad^2x=\lambda d^2y+\Omega^\alpha y_\alpha$, the tangential deformation is called conformally tangential, and theorems are also given for this case. *D. J. Struik* (Cambridge, Mass.).

Backes, F. Une autre forme de l'équation aux dérivées partielles du second ordre dont dépend le problème de la déformation des surfaces. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 452–457.

L'auteur ramène l'étude de la déformation à partir d'une surface donnée à celle de l'intégration d'une équation aux dérivées partielles de Monge-Ampère, cette dernière ayant la particularité d'être linéaire par rapport aux dérivées partielles du second ordre. Pour ce faire, il détermine les coefficients de la seconde forme quadratique fondamentale à l'aide des équations de Gauss et de Codazzi, en supposant toutefois connus ceux de la première.

F. Şemin (Istanbul).

Blanuša, Danilo. C^∞-isometric imbeddings of cylinders with hyperbolic metric in Euclidean 7-space. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 243–246. (Serbo-Croatian summary)

The two possible types of 2-dimensional hyperbolic cylinder are explicitly imbedded in Euclidean 7-space.

J. Schwartz (New York, N.Y.).

See also: Ordinary Differential Equations: Terracini. **Convex Domains, Integral Geometry:** Tsuji.

Manifolds, Connections

Devi Singh, Kamla. On generalised Riemann spaces. Riv. Mat. Univ. Parma **7** (1956), 125–138.

A space with a non-symmetric fundamental tensor was introduced by Einstein and later on dealt with by a great

number of mathematicians [cf. Schouten, Ricci-calculus, 2nd ed., Springer, Berlin, 1954, p. 179; MR 16, 521]. Some properties are derived in this paper. The publications after 1952, especially those of V. Hlavaty, seem to be unknown to the author. *J. A. Schouten* (Epe).

★ **Duff, G. F. D. Various classes of harmonic forms. Algebraic geometry and topology. A symposium in honor of S. Lefschetz,** pp. 129–138. Princeton University Press, Princeton, N. J., 1957. $7.50.

Let $M+B$ be a compact C^∞ Riemannian "manifold with boundary", and let t and n be the operators replacing a differential form by its tangential, resp. normal, components on the boundary B. A harmonic form ϕ is one for which $\Delta\phi=0$ ($\Delta=d\delta+\delta d$), while a harmonic field satisfies the stronger condition $d\phi=\delta\phi=0$. The author lists three previously proved existence theorems for harmonic forms with prescribed boundary data [Canad. J. Math. **5** (1953), 57–80; **6** (1954), 427–440; MR **14**, 903; **17**, 474], and uses them to prove that there exist co-closed ($\delta\phi=0$) harmonic forms ϕ on M with either (1) prescribed (boundary) values of $t\phi$, or (2) prescribed "admissible" values of $t*d\phi$, thus establishing a conjecture of Tucker's. Using these and orthogonal decomposition, a simplified proof is given that there exists a unique harmonic field ϕ on M having prescribed "admissible" values of $t\phi$ and prescribed periods on $R_p(M, B)$ (=relative Betti number) given independent relative p-cycles. A "kind of dual" result is obtained as a corollary by examining the period matrix. Finally, (1) and (2) are strengthened to: there exist (1′) a unique co-derived (i.e., $\phi=\delta\psi$) harmonic form ϕ with prescribed values of $t\phi$, and (2′) a non-unique co-derived harmonic form with prescribed "admissible" values of $t*d\phi$. *M. Gaffney* (New York, N.Y.).

Nakae, Tatuo. Dimensional differentiation of harmonic tensors for variations of Riemannian metric. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 43–53.

The concept is introduced of the abstract dimension of tensors and scalars in a properly Riemannian space of class C^∞ in which the dimension of the metric $[g_{ij}dx^idx^j]$ is equal to 2. If here $[dx^i]=0$, $[g_{ij}]=2$, then the dimension is called absolute, and if $[dx^i]=1$, $[g_{ij}]=0$, then relative. The dimensional derivative of the tensor $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ of weight w is defined by the formula

$$\delta A^{i_1\cdots i_p}_{j_1\cdots j_q}=vA^{i_1\cdots i_p}_{j_1\cdots j_q}+\sum_{k=1}^{p}\omega^{i_k}_{\alpha}A^{i_1\cdots\alpha\cdots i_p}_{j_1\cdots j_q}-\sum_{k=1}^{q}\omega^{\alpha}_{j_k}A^{i_1\cdots i_p}_{j_1\cdots\alpha\cdots j_q}+(w+\alpha/n)\omega A^{i_1\cdots i_p}_{j_1\cdots j_q},$$

where vA denotes the variation of the tensor A with respect to some parameter t, and $\omega_{ij}=\frac{1}{2}vg_{ij}$, $\omega=g^{ij}\omega_{ij}$, $\alpha=[A]_{\text{rel}}$. The author obtains estimates of the dimensions of the derivative of the characteristic roots of an elliptic differential operator and proves that, if $G(x, y)$ is the Green's tensor of the corresponding equation, then δG is regular for all points of the domain D and $\delta G=0$ if x is on the boundary of the domain. An expression is also obtained for δG in terms of the covariant derivatives of G. A connection is established between the abstract relative dimension and the dimensions of the physical magnitude of an electromagnetic field.

G. I. Kručkovič (RŽMat **1956**, no. 9080).

Nakae, Tatuo. The local and global covariant variations of differential forms under an infinitesimal conformal transformation. J. Math. Soc. Japan **9** (1957), 20–37.

Let $\mathfrak{M}$ be an n-dimensional orientable Riemannian space of C^∞, $\mathfrak{D}$ be a bounded domain in $\mathfrak{M}$ with boundary $\mathfrak{B}$ which is the union of a finite number of disjoint closed manifolds of C^∞. For forms α, β defined on the neighborhood of $\mathfrak{D}+\mathfrak{B}$, we use the usual notations $*\alpha$, $\alpha\wedge\beta$, $d\alpha$, $\delta\alpha$, $\Delta\alpha=(d\delta+\delta d)\alpha$, $(\alpha\cdot\beta)=\int_{\mathfrak{D}}\alpha\wedge*\beta$, $\{\alpha\cdot\beta\}=\oint_{\mathfrak{B}}\alpha\wedge*\beta$. We denote by N^i the unit normal to B and put $\nu=N_idx^i$, $\perp\alpha=\nu\wedge\alpha$, $\top\alpha=*\nu\nu\alpha$, where $*(\alpha\wedge\beta)=*\beta\nu*\alpha$. We shall call a form α absolutely exact or closed [co-exact or co-closed] when $d\alpha=0$ [$\delta\alpha=0$], absolutely derived or homologous to zero [co-derived or co-homologous to zero] when $\alpha=d\beta$ [$\alpha=\delta\beta$], relative [co-relative] when $\perp\alpha=0$ [$\top\alpha=0$], relatively exact [relatively co-exact] when $\perp\alpha=0$, $d\alpha=0$ [$\top\alpha=0$, $\delta\alpha=0$], and relatively derived [relatively co-derived] when $\alpha=d\beta$, $\perp\beta=0$ [$\alpha=\delta\beta$, $\top\beta=0$]. We denote the set of all p-forms by $\mathfrak{C}_a{}^p$, the set of all relative [co-relative] p-forms by $\mathfrak{C}_r{}^p$ [$\overline{\mathfrak{C}}_r{}^p$], the set of all absolute exact [co-exact] p-forms by $\mathfrak{Z}_a{}^p$ [$\overline{\mathfrak{Z}}_a{}^p$], the set of all absolute derived [co-derived] p-forms by $\mathfrak{B}_a{}^p$ [$\overline{\mathfrak{B}}_a{}^p$], the set of all relative exact [co-exact] p-forms by $\mathfrak{Z}_r{}^p$[$\overline{\mathfrak{Z}}_r{}^p$], the set of all relative derived [co-derived] p-forms by $\mathfrak{B}_r{}^p$ [$\overline{\mathfrak{B}}_r{}^p$], and put $\mathfrak{Z}_a{}^p\cap\mathfrak{Z}_r{}^p=\mathfrak{H}_a{}^p$, $\overline{\mathfrak{Z}}_r{}^p\cap\mathfrak{Z}_a{}^p=\overline{\mathfrak{H}}_r{}^p$, $\mathfrak{Z}_r{}^p\cap\mathfrak{Z}_a{}^p=\mathfrak{H}_r{}^p$, $\overline{\mathfrak{Z}}_a{}^p\cap\mathfrak{Z}_r{}^p=\overline{\mathfrak{H}}_a{}^p$. A p-form α in $\mathfrak{H}_r{}^p=\overline{\mathfrak{H}}_a{}^p$ is called a relative harmonic or absolute co-harmonic form and is characterized by $d\alpha=0$, $\perp\alpha=0$, $\delta\alpha=0$, which is equivalent to $\Delta\alpha=0$, $\perp\alpha=0$, $\perp\delta\alpha=0$. A p-form α in $\mathfrak{H}_a{}^p=\overline{\mathfrak{H}}_r{}^p$ is called an absolute harmonic or relative co-harmonic form and is characterized by $\delta\alpha=0$, $\top\alpha=0$, $d\alpha=0$, which is equivalent to $\Delta\alpha=0$, $\top\alpha=0$, $\top d\alpha=0$.

The author states first the theorems of Conner: (1) The equations $\Delta\beta=\alpha$, $\perp\beta=0$, $\perp\delta\beta=0$ are consistent if and only if α is orthogonal to the space $\mathfrak{H}_r$. The equations $\Delta\beta=\alpha$, $\top\beta=0$, $\top d\beta=0$ are consistent if and only if α is orthogonal to the space $\mathfrak{H}_a$. (2) $\mathfrak{C}_a=\mathfrak{B}_r\oplus\mathfrak{H}_r\oplus\overline{\mathfrak{B}}_a$, $\mathfrak{C}_a=\mathfrak{B}_a\oplus\mathfrak{H}_a\oplus\overline{\mathfrak{B}}_r$. Let $H\alpha$, $K\alpha$ and $L\alpha$ be the orthogonal projections of α into $\mathfrak{H}_r$, $\mathfrak{B}_r$ and $\overline{\mathfrak{B}}_a$ respectively. The author proves: (3) $L\alpha$ is characterized by the equations $L\alpha=\delta\xi$, $d\delta\xi=d\alpha$, $\perp\delta\xi=\perp\alpha$. (4) $K\alpha$ is characterized by the equations $K\alpha=d\xi$, $\delta d\xi=\delta\alpha$, $\perp\xi=0$. The author then considers an infinitesimal conformal transformation $vg_{ij}=2\mu g_{ij}$ of the metric tensor g_{ij}, and defines an operator δ_l by the formula $\delta_l\alpha_p=v\alpha_p+M\alpha_p$, where $M\alpha_p=(\frac{1}{2}n-p)\mu\alpha_p$. We call $\delta_l\alpha$ the local covariant variation of a form α. The author gives in his fifth theorem the formulas giving the local variations of the operators, H, K and L. He then defines the global covariant variation δ_g by the formula $\delta_g\alpha=H\delta_lH\alpha+K\delta_lK\alpha+L\delta_lL\alpha$ and proves the following. (6) The operator δ_g is commutative with the operators H, K and L. (7) If α is in C_r (i.e., if α remains in $\mathfrak{C}_r$ during the transformation), then $\delta_g\alpha$ is also in $\mathfrak{C}_r$. (8) If α is in $\mathfrak{H}_r$, $\mathfrak{B}_r$, $\overline{\mathfrak{B}}_a$, then $\delta_g\alpha$ is in $\mathfrak{H}_r$, $\mathfrak{B}_r$, $\overline{\mathfrak{B}}_a$ respectively. (9) $v(\alpha\cdot\beta)=(\delta_g\alpha\cdot\beta)+(\alpha\cdot\delta_g\beta)$. (10) The global covariant variation of the Green operator $G(x, y)$ is given by $\hat{G}\alpha=G\hat{\Delta}G\alpha-\hat{G}\delta G\alpha$, where the dot denotes δ_g and $\hat{G}\alpha=(G\alpha)^{\cdot}-G\dot{\alpha}$, $\hat{\Delta}\alpha=(KMd\delta+d\delta MK-LM\delta d-\delta dML-2dM\delta+2\delta Md)\alpha$, $\delta\alpha=(-LM\delta+\delta MK)$, and $\hat{G}\alpha=\{\alpha(y)\cdot G(x, y)\}$. In the last section the author considers an r-parameter group of conformal transformations generated by $\xi_a{}^i$. Let α_λ be a set of independent generators of H_r, and put $\delta_g(\xi)\alpha_\lambda=K_\lambda{}^\mu\alpha_\mu$, $R_\lambda{}^\nu{}_{ab}=K_\lambda{}^\mu{}_aK_\mu{}^\nu{}_b-K_\lambda{}^\mu{}_bK_\mu{}^\nu{}_a$. The author's last theorem states that the covariant derivative of the curvature tensor $R_\lambda{}^\nu{}_{ab}$ for a conformal transformation group is zero. *K. Yano* (Tokyo).

Elianu, Jean. Sur les courants polyharmoniques. C. R. Acad. Sci. Paris **245** (1957), 486–488.

Let V be a C^∞ (analytic) Riemannian manifold; denote

by Δ^m the mth iterated Laplacian on V. The author proves that if T is a courant on V such that $\Delta^m T$ is equal in a domain D to a C^∞ (analytic) form, then T is itself equal to a C^∞ (analytic) form in D. The proof is an easy induction on the known case $m=1$. In case V is compact, the equation $\Delta^m S=T$ (S and T are courants) has a solution S if and only if T is orthogonal to all harmonic forms; in that case, there is a unique solution Σ orthogonal to all harmonic forms, and every solution $S=\Sigma+h$, where h is harmonic. In particular, if $T=0$, then S is harmonic. The proof uses de Rham's description of the action of Δ on the space of courants, and induction.

L. Ehrenpreis (Waltham, Mass.).

Aragnol, André. Espace des tenseurs de type adjoint associé à un espace fibré principal. Sous-espaces. Homomorphismes. C. R. Acad. Sci. Paris **244** (1957), 2683–2686.

A consideration of vertical plane distributions (normal fields) in a principal fiber bundle with a connection leads to the concept of stability for these distributions, by associating them with Lie algebra valued forms. The intersection of all normal fields is the field of holonomy. Ambrose and Singer's theorem on holonomy [Trans. Amer. Math. Soc. **75** (1953), 428–443; MR **16**, 172] follows automatically. Further applications are made to connections on subspaces, and to bundle homomorphisms.

A. Nijenhuis (Seattle, Wash.).

Aragnol, André. Théorie des connexions. C. R. Acad. Sci. Paris **244** (1957), 437–440.

The author formulates the theory of connections in the language of sheaves, and derives some theorems on local and infinitesimal holonomy groups, some of which resemble results of Nijenhuis [Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 233–240, 241–249; **57** (1954), 17–25; MR **16**, 171, 172] as generalized by Ozeki [Nagoya Math. J. **10** (1956), 105–123; MR **18**, 232]. The small number of pages available causes the definitions to be more numerous than the theorems. The reviewer is of the opinion that both author and reader would greatly benefit from a more elaborate exposition.

A. Nijenhuis (Seattle, Wash.).

★**Kobayashi, Shôshichi. Le groupe des transformations qui laissent invariant le parallélisme.** Colloque de topologie de Strasbourg, 1954–1955, Institut de Mathématique, Université de Strasbourg. 5 pp.

A proof is given of the following theorem: If V is a differentiable manifold with a given absolute parallelism, then the group of transformations of V preserving the parallelism is a Lie group. As observed by the author, this theorem is an immediate consequence of the Myers-Steenrod Theorem that the group of isometric transformations of a Riemannian manifold is a Lie group. However, the author's method also permits imbedding the group regularly as a submanifold of V.

S. T. Hu.

Rapcsák, A. Eine neue Charakterisierung Finslerscher Räume skalarer und konstanter Krümmung und projektiv-ebene Räume. Acta Math. Acad. Sci. Hungar. **8** (1957), 1–18.

Guided by well-known theorems on Riemannian spaces of constant curvature, the author seeks generalisations of such theorems applicable to Finsler spaces F_n. Three types of "hyperplanes" are defined as follows. First kind: these are hypersurfaces whose geodesics (with respect to the induced metric) are also geodesics of F_n, and conversely. Second kind: for these the same condition is satisfied by the quasi-geodesics [for the definition of these curves see O. Varga, C. R. Premier Congrès Math. Hongrois, 1950, Akad. Kiadó, Budapest, 1952, pp. 131–162; MR **15**, 160]. Third kind: hypersurfaces whose normals result from each other by parallel displacement. Theorem 1: In order that it be possible to construct a hyperplane of the first kind for each line-element of F_n (with given normal direction) it is necessary and sufficient that F_n be projectively flat [L. Berwald, Ann. of Math. (2) **48** (1947), 755–781; MR **9**, 207]. Theorem 2: The same is true for hyperplanes of the second kind if, in addition, the tensor $A_{\alpha\beta\gamma 10}$ vanishes identically in F_n. Theorem 3: The same is true for hyperplanes of the third kind if and only if F_n is a Riemannian space of constant curvature.

H. Rund (Durban).

Bucur, I. Sur une propriété globale des lignes géodésiques d'un espace de Finsler. Com. Acad. R. P. Romîne **5** (1955), 965–968. (Romanian. Russian and French summaries)

The author extends to compact Finsler spaces the theorem of Serre that between any two points of a compact Riemannian manifold there pass an infinite number of geodesics.

J. M. Danskin (Princeton, N.J.).

★**Yano, Kentaro. The theory of Lie derivatives and its applications.** North-Holland Publishing Co., Amsterdam; P. Noordhoff Ltd., Groningen; Interscience Publishers Inc., New York, 1957. x+299 pp. $8.75.

This is a comprehensive treatise of the theory of Lie derivatives. Using a unified method, the author establishes most of the known results on groups of local automorphisms of spaces with geometrical objects. When the space is compact Riemannian or compact pseudo-Kählerian, global properties are also discussed. The proofs are given in detail, and the presentation is clear.

In Chapter I, the author introduces the notion of Lie derivatives by means of some particular cases. Chapter II gives the definition of Lie derivative of a geometrical object with respect to an infinitesimal transformation while Chapter III deals with the general properties of groups (or more precisely, Lie algebras) of infinitesimal transformations leaving invariant a linear geometrical object.

Chapters IV–VIII are concerned with groups of automorphisms of spaces with the classical structures (i.e., Riemannian, Finsler, affinely connected, projectively connected, conformally connected spaces, as well as spaces of paths and k-spreads). Among other results, the author treats in great detail those spaces which admit groups of automorphisms of sufficiently high dimension. Results of Egorov, Muto, Vranceanu and the author are proved in a systematic way. The local structures of such spaces are completely determined, and when the space is Riemannian the topological structure is also given.

Chapter IX is devoted to global properties of compact Riemannian spaces. Some results of Bochner, the author and Lichnerowicz are established. Chapter X deals with almost complex spaces. It opens with a brief exposition of almost complex structure, Nijenhuis tensor (also called Eckmann-Frölicher tensor) and the various linear connections associated with an almost Hermitian metric. The author then restricts himself to pseudo-Kählerian spaces, and discusses their curvature and pseudo-analytical vector fields. Two geometrical characterizations of pseudo-Kählerian spaces with constant holomorphic

curvature are given. {Reviewer's note: This book was written before the publication of the theorem of A. Newlander and L. Nirenberg [see the paper reviewed below] which states that each pseudo-complex space is complex.}

This book concludes with an appendix containing many results, which appeared in the last few years, on groups of automorphisms of spaces with geometrical structure. Most of them are global in nature. *H. C. Wang.*

See also: Analytic Theory of Numbers: Selberg. **Differential Geometry:** Lagally. **Mechanics of Particles and Systems:** Storchi.

Complex Manifolds

Newlander, A.; and Nirenberg, L. Complex analytic coordinates in almost complex manifolds. Ann. of Math. (2) **65** (1957), 391–404.

This paper solves the geometrical problem of proving that a sufficiently differentiable, almost complex manifold is a complex manifold, provided certain "integrability" conditions are satisfied. The problem, in an analytical formulation, is reduced to that of finding n independent (complex-valued) solutions $\zeta^1, \cdots, \zeta^n$ of an overdetermined system of partial differential equations of the form (1) $L_j w \equiv \bar\partial_j w - a_j{}^k \partial_k w = 0$ $(j=1, \cdots, n)$ when the L_j commute. Here, $\partial_j = \partial/\partial z^j$ and $\bar\partial_j = \partial/\partial \bar z^j$, where the z^i represent complex coordinates and the $\bar z^i$ their complex conjugates; the condition $a_j{}^i = 0$ for $z^1 = \cdots = z^n = 0$ is assumed. An arbitrary solution w of (1) would be an analytic function of these ζ^i; hence, in particular, w would satisfy the Cauchy-Riemann equations (2) $\bar d_j w = 0$, where $\bar d_j = \partial/\partial \bar\zeta^j$ (we shall also write $d_j = \partial/\partial \zeta^j$). This shows, in view of the presumed equalities $\bar d_j w = \partial_k w \cdot \bar d_j z^k + \bar\partial_k w \cdot \bar d_j \bar z^k = \partial_k w (\bar d_j z^k + a_m{}^k \bar d_j \bar z^m)$, obtained from (1), that solving (1) is equivalent to solving the system of equations (3) $\bar d_j z^k + a_m{}^k \bar d_j \bar z^m = 0$. The latter system is simpler than (1) in that but one kind of differentiation enters into each individual equation of the system, and the system is solved by an ingenious iterative procedure based on the fact that an equation of the form $\bar d_j z = f$ is inverted by a double integral in the ζ^j plane. *A. Douglis.*

Vesentini, Edoardo. Sui punti stazionari di forme differenziali meromorfe sopra una varietà complessa compatta. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 486–494.

The author considers, on a complex compact analytic manifold M, certain systems of meromorphic differential forms having a locus of stationary points of maximum possible dimension, and obtains an expression for the homology class of this locus in terms of the classes of the polar locus of the forms and the Chern classes of M. The results generalise theorems obtained, for algebraic varieties, by Eger and Hodge. *J. A. Todd.*

Vesentini, Edoardo. Sugli jacobiani di funzioni meromorfe sopra una varietà complessa compatta. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 428–434 (1956).

Summary of results expounded in detail elsewhere [Ann. Mat. Pura Appl. (4) **42** (1956), 325–379; MR **18**, 763]. *J. A. Todd* (Cambridge, England).

Singh, K. D. Sous-espaces d'une variété kählérienne. Bull. Sci. Math. (2) **81** (1957), 21–29.

The author gives several properties of submanifolds of a Kaehler manifold, among which are the following. Every submanifold of a Kaehler manifold is Kaehler. The mean curvature of any such submanifold is zero; the divergence of a unit normal vector to such a submanifold is zero. A submanifold, which has the property that each point is an umbilic of the ambient space, is geodesic and has zero normal curvature in every direction. *R. C. Gunning* (Princeton, N.J.).

★ **Séminaire H. Cartan de l'Ecole Normale Supérieure, 1953/1954. Fonctions automorphes. Exposés de J.-P. Serre, 24-5-54 et 14-6-54.** 25 pp.

This last lecture of the 1953/54 notes [for Lectures XVI–XIX see MR **18**, 69] is an application of the techniques introduced in the earlier lectures, and in the 1951/52 notes [MR **16**, 233], to some problems concerning automorphic functions of several complex variables. Let X be a complex manifold, G a group of analytic automorphisms of X satisfying the discontinuity conditions of lecture XII [cf. also H. Cartan, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, Princeton, 1957, pp. 90–102; MR **18**, 823], $Y = X/G$, and $\pi: X \to Y$ the canonical projection map; Y is then a general analytic space in the sense of Cartan [loc. cit.]. A factor of automorphy J_g is a correspondence associating to each $g \in G$ a holomorphic, non-zero function $J_g(z)$ on X such that $J_{gg'}(z) = J_g(g'z) J_{g'}(z)$. It is noted that locally all such factors are essentially merely characters of the isotropy subgroups. For a given factor J_g, associate to each open set $U \subset Y$ the family $\mathscr{A}(J)_U$ of functions $f(z)$ holomorphic in $\pi^{-1}(U)$ and satisfying $f(gz) = J_g(z) f(z)$; the sheaf $\mathscr{A}(J)$ defined by the collection $\mathscr{A}(J)_U$ is shown to be a coherent analytic sheaf over the analytic space Y, and is the principal tool in the applications. Note that the integral powers $J_g(z)^m$ are also factors of automorphy; the sheaf they define will be denoted by $\mathscr{A}(J^m)$.

The first applications are to the case in which X is a bounded subdomain of the complex affine space, and $J_g(z)$ is the complex Jacobian of the transformation g. As is well known [see Lecture XV, or Cartan, loc. cit.], Y can be imbedded as a normal analytic subvariety of a complex projective space P^r by means of a number of automorphic forms (elements of $H^0(Y, \mathscr{A}(J^{-m}))$); upon this imbedding any coherent analytic sheaf over Y can be extended to a coherent analytic sheaf over the ambient space P^r, having trivial stalk at all points not in Y, and the special results developed by the author in Lectures XVIII–XIX [MR **18**, 69] for coherent analytic sheaves over projective spaces are applicable. For the particular sheaves $\mathscr{A}(J^{-m})$, these techniques are used to prove such results as that all automorphic forms of sufficiently high weight are Poincaré series and that the graded algebra of automorphid forms, and every ideal therein, is finitely generated; anc to obtain more explicit formulas for the dimension of the linear space of automorphic forms of weight m as a function of m. As a further application, if E is an analytic subvariety of X stable under G, then G induces a group of analytic automorphisms on E, and factors of automorphy $J_g(z)^m$ define factors of automorphy on E by restriction. It is proved that the corresponding sheaves $\mathscr{A}_E(J^m)$ are coherent analytic sheaves on E, and as a consequence that for sufficiently high weights m all cross-sections in $H^0(E, \mathscr{A}_E(J^{-m}))$ are restrictions to E of automorphic forms on X. Most of the results in both applications had been obtained earlier by Hervé for complex dimension 2 [Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 277–302; MR **14**, 633]. *R. C. Gunning* (Princeton, N.J.).

★**Kodaira, K.; and Spencer, D. C. On the variation of almost-complex structure.** Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 139–150. Princeton University Press, Princeton, N. J., 1957. $7.50.

This relatively short paper contains a number of very interesting results. Some are new; some proved in a better way. The main or final results are the two theorems: (2.1) Every almost Kähler manifold possesses an integrable structure; (4.1) The dimensions $h^{r,s}(t)$ of the spaces $\mathcal{H}^{r,s}(t)$ of harmonic forms of almost-complex Hermitian structures depending on t (a point of a parameter manifold), with respect to almost complex structures $\partial(t)$, are upper semi-continuous functions of t. The second theorem is subsequently generalized for groups $H^s(M, \Omega^r(F(t)))$ with respect to line bundles in a given complex manifold, depending on t. Applications to algebraic varieties are indicated; and some examples are provided to show that the functions $h^{r,s}(t)$ can have discontinuities.

All theorems are proved by methods utilizing harmonic analysis. As to theorem (2.1), the authors' "almost complex Kähler manifold" is not what Yano calls "almost Kähler manifold", but is defined by the reality of $\Delta = 2(b\bar{\partial}+\bar{\partial}b)$; and the theorem does not state that the structure is actually Kählerian. *A. Nijenhuis.*

See also: Functions of Complex Variables: Morita.

Algebraic Geometry

Svoboda, Karel. Contribution à la théorie d'une courbe normale d'un espace à quatre dimensions. Časopis Pěst. Mat. **82** (1957), 301–307. (Czech. Russian and French summaries)

In the projective space P_4 of four dimensions let J be a cubic hypersurface which has as double curve C a rational normal curve of fourth degree [as introduced by H. G. Telling, The rational quartic curve in space of three and four dimensions, Cambridge, 1936]. The author studies a correspondence between two systems of straight lines on J, the binary lines, that is, the family of bisecants (chords) of C, and the unary lines, forming the system of the leading lines of the involutions on C. There are three lines through every point of J, not on C; one binary, two unary. The binary line coincides with one of the unary lines if and only if their common point lies on a tangent to C. In a synthetic way the author derives a mapping of these straight lines into a plane; this mapping has first been studied by R. K. Wakerling [Duke Math. J. **14** (1947), 795–802; MR **9**, 247]. He then proves that the correspondence between the two ruled surfaces, generated by the line systems, has as image in the plane the polarity with respect to the conic section which represents the surface of the tangents of C. Hence follow certain properties of the correspondence between the ruled surfaces in P_4. *H. Schwerdtfeger* (Montreal, P.Q.).

Lang, Serge. Sur les séries L d'une variété algébrique. Bull. Soc. Math. France **84** (1956), 385–407.

L'auteur rapporte deux compléments à la "théorie du corps de classes" qu'il a développée dans son mémoire dédié à Artin [Ann. of Math. (2) **64** (1956), 285–325; MR **18**, 672].

Il étudie d'abord les zéros et les pôles des séries L; soient k un corps fini à q éléments, K un corps de fonctions algébriques à r variables sur k, E une extension galoisienne finie de K, et M une représentation linéaire de son groupe de Galois. Si l'on choisit un schéma ("modèle") U de K, non singulier, et tel que E/K soit non ramifié sur U, on peut définir (loc. cit.) la substitution de Frobenius $\sigma_\mathfrak{p}$ d'un cycle premier $\mathfrak{p}$ de dimension zéro de U; d'où, par la formule d'Artin, une fonction L:

$$L_{M,U}(t) = \prod_{\mathfrak{p}} 1/\det(1 - t^{\deg(\mathfrak{p})} M(\sigma_\mathfrak{p})).$$

Supposons alors que la représentation M soit irréductible et non triviale, et que k soit algébriquement fermé dans E (c'est le cas "géométrique" — il est dommage que le cas général ne soit pas traité). L'auteur démontre alors que la fonction $L_{M,U}$ est holomorphe et partout non nulle dans le disque $|t| < q^{-(r-\frac{1}{2})}$. La démonstration, assez délicate, consiste à ramener la question au cas des courbes au moyen de sections planes convenable; c'est là un procédé voisin de celui qu'avait déjà employé l'auteur pour l'étude de la fonction zêta [S. Lang et A. Weil, Amer. J. Math. **72** (1954), 819–827; MR **16**, 398; voir aussi L. B. Nisnevic, Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 17–20; MR **16**, 743].

Dans une seconde partie, l'auteur étend sa théorie du corps de classes à de nouveaux types d'extensions. Si $\alpha: U \to G$ est une application rationnelle de U dans un groupe algébrique commutatif G (α, U et G étant définis sur k), on dit que U et G sont bien adaptés pour α si α ne peut pas se factoriser en $U \to G' \to G$, où $G' \to G$ est une isogénie non triviale de G (définie sur une extension k' de k). Dans ce cas, l'image réciproque ("pull-back") par α d'une isogénie séparable $G' \to G$ de degré n est un revêtement abélien irréductible et de degré n de U; les revêtements $V \to U$ qui peuvent être obtenus de cette manière (sur une extension de k) sont dits "de type α". L'auteur indique (le plus souvent sans démonstration) comment les résultats de son mémoire cité plus haut s'étendent aux revêtements de type α (qui remplacent ceux "du type d'Albanese"); le principal intérêt de cette extension provient de ce que tout revêtement abélien est de type α pour un α convenable (appliquer Kummer ou Artin-Schreier-Witt). Dans le cas où U est une courbe, on peut prendre pour G une jacobienne généralisée de U (au sens de Rosenlicht); les relations de ces jacobiennes avec le groupe des classes d'idèles redonnent alors la théorie du corps de classes sous sa forme usuelle, et cela sans qu'il ait été nécessaire de faire aucun calcul d'indices, ni de démontrer aucune de ces pénibles "inégalités" qui déparent le cas des corps de nombres. *J.-P. Serre* (Paris).

Todd, J. A. Canonical systems on algebraic varieties. Bol. Soc. Mat. Mexicana (2) **2** (1957), 26–44.

The present paper gives a very clear and comprehensive survey of the theory of canonical systems from the algebraic-geometric point of view. Moreover, it emphasizes the aspects which are in close contact with modern topological developments, adding some new contributions and critical remarks, and it contains an ample bibliography on the subject (to which B. Segre, Proc. Internat. Congress Math., Amsterdam, 1954, v. III, Noordhoff, Groningen, 1956, pp. 497–513 [MR **19**, 319] could be added).

After a few introductory explanations on equivalence relations among (virtual) subvarieties of an algebraic variety, the initial work by Severi, B. Segre, Eger and Todd on canonical systems is summarized, and two recent approaches to the theory by Segre are given in detail (without, however, entering into the many-sided applica-

tions to intersections and other problems). One of these approaches — topological in character — is by means of dilatations, and leads the author to an algebraic-geometric interpretation of the formula of Hirsch. Then the intimate relations between the other approach and the theory of Chern cohomology classes is explained, and the behaviour of the canonical systems under birational transformations is investigated. Thus, by using a result by Hirzebruch on Chern classes, it is shown that an algebraic equivalence answering the last question holds, if the equivalence relation is replaced in it by the weaker one of homology (the equivalence was proved by Todd and Segre in special cases, and conjectured in general by Todd in an earlier work). The paper ends by formulating two interesting conjectures about numerical invariants of an algebraic variety. *B. Segre* (Rome).

Segre, Beniamino. Corrispondenze birazionali e topologia di varietà algebriche. Ann. Mat. Pura Appl. (4) **43** (1957), 1–23.

The first part of this paper deals with properties of the special type of birational correspondence known as a dilatation, and, more particularly, with the effects of a dilatation on the base numbers for algebraic cycles of various dimensions. It is proved that any birational transformation between two non-singular algebraic surfaces can be expressed as the product of a finite number of dilatations and their inverses, and a biregular transformation.

The second part of the paper is concerned with the homology groups of the topological space which is the Riemannian of $V-M$, where V is an algebraic variety of dimension d and M a subvariety (possibly reducible and impure) of dimension $c<d$. The topological dimension of $V-M$ is $2d$, and the homology and torsion groups for dimension less than $2d-2c-1$ are the same as those of V. The author finds a base for homology on $V-M$ for dimension $2d-2c-1$, giving a system of generators and relations. The generators are those of V together with a set of cycles, one for each component of M of maximum dimension, which are (referred to a suitable Riemannian metric) the boundaries of geodesic $(2d-2c)$-spheres drawn with centre at a general point of each component in M, lying in V and normal to M. The relations between the generators involve the linking coefficients of M with the basic cycles on V. The paper concludes with some applications to birational transformations of one variety into another. *J. A. Todd* (Cambridge, England).

Rosenlicht, Maxwell. A universal mapping property of generalized jacobian varieties. Ann. of Math. (2) **66** (1957), 80–88.

Soient C une courbe complète non singulière, f une application rationnelle de C dans un groupe algébrique commutatif G, et $P_1, \cdots, P_s$ les points de C où f n'est pas régulière; il existe alors une relation d'équivalence sur C [Rosenlicht, Ann. of Math. (2) **56** (1952), 169–191; MR **14**, 80] admettant les P_i pour places et telle que, si on note J la jacobienne généralisée correspondante [Rosenlicht, ibid. **59** (1954), 505–530; MR **15**, 823] et g l'application canonique de C dans J, alors il existe un homomorphisme rationnel h de J dans G et un point a de G (uniquement déterminés) tels que $f(P)=h(g(P))+a$ pour $P \in C$; l'auteur donne aussi un énoncé plus précis, tenant compte des corps de définition et de la présence sur C de points k-simples mais pas absolument simples. La démonstration traite d'abord le cas où G est une extension d'un produit de droites multiplicatives par une variété abélienne. Puis on passe en caractéristique $p \neq 0$, où le th. de Chevalley sur la structure des groupes algébriques [Rosenlicht, Amer. J. Math. **78** (1956), 401–443; MR **18**, 514] permet de se réduire au cas où G est un groupe de matrices unipotentes mises sous forme triangulaire; ce cas est alors traité par des méthodes assez explicites. En caractéristique 0 l'auteur donne une démonstration directe assez simple utilisant les formes différentielles invariantes sur G; ceci permet de simplifier notablement, en caractéristique 0, la démonstration du th. de Chevalley donnée par l'auteur (loc. cit.). *P. Samuel* (Clermont-Ferrand).

Burau, W. La projezione stereografica e le sue generalizzazioni. Rend. Sem. Mat. Messina **1** (1955), 88–93.

This paper gives a wide generalisation of the stereographic projection of a general quadric into a hyperplane. It is stated to be valid for all self-dual varieties, but in fact is proved for (i) Veronesians, (ii) Segre products of linear spaces, (iii) Grassmannians, (iv) the minimum model of either system of generating S_k's on a general quadric in S_{2k+1}, and (v) the minimum model of the system of self-corresponding S_k's in a general null polarity in S_{2k+1}. For each of these varieties, say V_r, it is shown that the intersection of a fixed tangent S_r with a general hyperplane of the dual figure does not lie in any other hyperplane of the dual figure, and hence that the projection into S_r from the dual of a tangent S_r is one-one. *P. Du Val.*

Togliatti, Eugenio Giuseppe. Alcune vedute recenti sulle trasformazioni Cremoniane. Confer. Sem. Mat. Univ. Bari no. **19** (1956), 12 pp.

A survey of recent results on Cremona transformations between two planes: the interest is focused on the representation of a point-transformation on C. Segre's manifold and on the new tools provided by differential geometry. *E. Bompiani* (Rome).

See also: Complex Manifolds: Kodaira and Spencer.

NUMERICAL ANALYSIS

Numerical Methods

★**Modern computing methods.** Notes on applied science, no. 16, National Physical Laboratory, Teddington, England. Her Majesty's Stationery Office, London, 1957. vi + 129 pp. $1.89.

Excellent brief resumé. The table of contents includes: linear equations and matrices; roots of polynomial equations; latent roots of matrices; finite-difference methods; ordinary diffrential equations; hyperbolic, elliptic, and parabolic differential equations; relaxation methods; tabulation and computation of mathematical functions. Appendices give a bibliography on each of the several topics; a brief description of the Deuce; and a description of a mechanical differential analyzer. Attention is paid to computing by hand as well as by machine, with injunctions for each. *A. S. Householder* (Oak Ridge, Tenn.).

★**Willers, Friedrich Adolf. Methoden der praktischen Analysis. 3te Aufl.** Walter de Gruyter & Co., Berlin, 1957. 429 pp. DM 28.00.

The first edition of this book appeared in 1928, an

English translation in 1948 [Dover, New York; MR 10, 404], and the second edition was published in 1950. The preface to the second edition remarks that no consideration is given to the use of large computing machines since such machines are not available in Germany. This was true at that time.

In the present edition the text has been expanded by 18 pages, mainly in extending the discussion of the method of Graeffe, and a short description of methods of relaxation in the solution of linear systems. A few references to more recent literature have been added. With a few other minor exceptions the two editions are word for word identical; and illustrations are unchanged, including Fig. 11, "Vollautomatische Rechenmaschine," and Fig. 12, "Sprossenradmaschine für Handbetrieb."

A. S. Householder (Oak Ridge, Tenn.).

★ **Хухриков, С. С. [Huhrikov, S. S.] Приближенный численный метод расчета переходных процессов в линейных и нелинейных системах: метод рекуррентных формул. [Approximate numerical method for computation of transition processes. Method of recursive formulas.]** Gosudarstv. Izdat. Oboronnoĭ Promyslennosti, Moscow, 1957. 59 pp. 2.30 rubles.

The method allows an explicit step-by-step approximation of the transient amplitude at times $t=n\tau$ ($n=1, 2, \cdots$) from the initial conditions, with τ reasonably $\ll$ (guessed smallest time constant characteristic of transient). It is based on the approximation

$$f((m+1)\tau)\approx(-1)^m(m!\tau^{m+1})^{-1}\cdot(d^mF(p)/dp^m)_{p=\tau^{-1}},$$

where $f(t)$-transient and $F(p)$-its Laplace mate, justified for $m^2\gg 1$. In the present instance it is applied from $m=1$ on, which introduces considerable error unless τ is fairly small. In spite of the lack of legitimacy and of the unavailability of error estimates, the method seems of value for obtaining coarse engineering approximations because of its ease of application and great flexibility. It is, in particular, applicable to the type of non-linear problems met in automatic control; in this case, the recurrency relations are solved graphically. Stability considerations are outside the scope. Several simple examples covering a wide range of electrical and electromechanical circuits, linear and non-linear, with lumped and distributed parameters are treated in some detail.

H. G. Baerwald (Cleveland, O.).

Muller, Mervin E. Some continuous Monte Carlo methods for the Dirichlet problem. Ann. Math. Statist. **27** (1956), 569–589.

The paper employs stochastic models which are Markovian processes to obtain solutions of the N-dimensional spherical, general spherical, and general domain Dirichlet problems. The method employs sampling of random walks and averaging of the first crossing values to obtain approximate solutions to the Dirichlet problem. The results are valid in all cases where the existence and uniqueness of the solution to the Dirichlet problem can be proved. The random walk is obtained by an iterated spherical process. One important fact established in the paper is that the computational effort involved in obtaining a solution within a given accuracy depends linearly on N. This is an essential advantage hoped for in the use of the Monte Carlo technique. In addition, errors due to truncating the variables (which are unavoidable since computing machines use only a finite number of digits) are estimated; the order of the number of steps required for obtaining a given approximation is estimated together with an evaluation of the variance. Finally, truncation of higher order is discussed and the errors can be diminished by higher order procedures utilizing the knowledge of the curvature of the boundary of the region for which the Dirichlet problem is to be solved.

S. M. Ulam (Los Alamos, N.Mex.).

★ **Metropolis, N. Phase shifts - middle squares - wave equations.** Symposium on Monte Carlo methods, University of Florida, 1954, pp. 29–36. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1956. $7.50.

The author reports on three problems studied by Monte Carlo methods at Los Alamos during 1953–54. The first problem is hydrogen scattering by pions; this is reduced to finding relative minima on a six-dimensional toroidal surface. The second problem concerns the iterative "midsquare method" proposed by von Neumann for generating random digits; the number and length of chains of distinct iterates are studied as functions of the number of digits used and the number base. Presented last is a Monte Carlo approach to the solution of the time-independent Schrödinger equation with general potential function; preliminary experiments in three dimensions are described. *M. A. Hyman* (Philadelphia, Pa.).

★ **Ремез, Е. Я. [Remez, E. Ya.] Общие вычислительные методы Чебышевского приближения. Задачи с линейно входящими вещественными параметрами. [General computation methods for Čebyšev approximation. Problems with real parameters entering linearly.]** Izdat. Akad. Nauk Ukrainsk. SSR. Kiev, 1957. 454 pp. 30.40 rubles.

This is the first volume to include between its two covers a fairly complete development of Chebyshev approximation, its theory and its practice. Such a book is long overdue. Techniques are illustrated by a number of special examples worked out in detail, and different methods are applied to the same problem by way of comparison. The elaborate chapter and section headings make the Table of Contents into a detailed outline that facilitates reference. Little is presupposed on the part of the reader except the most basic concepts of algebra and analysis. An appendix sets forth the theory of convex bodies as far as it is required in the main text.

These are weighty recommendations and there are no others. The language is turgid and often bombastic, the explanations prolix and wearisome. The pages are replete with footnotes, chips off the workbench, generally unnecessary and frequently distracting. The notation is cumbersome and confusing, rather suggestive of Chebyshev's own, to be expected in the early 19th century but inexcusable in the middle 20th. Nevertheless, most of the material is to be found elsewhere only in journals and hence it is an important book for the numerical analyst.

The book is in two parts, the first devoted to the "properties of the solution of the general Chebyshev problem," the second to "finite systems of inconsistent equations and the method of nets in Chebyshev approximation," there being four chapters in the first and three in the second part. Chapter I considers the one-dimensional problem, first with polynomials, then with systems of more general functions. Chapter II takes up inconsistent linear equations; Chapter III certain cases of approximation that can be solved "without a sequence of numerical approximations," including the case of minimiza-

tion of the relative error. And Chapter IV discusses "some methods and examples of approximations connected with Chebyshev requirements."

The title of Chapter V, the first of Part II, is misleading, since it refers to "weighted quadratic approximation," whereas the chapter deals mainly with mean mth power approximation, regarding m as large. The next chapter describes a method of steepest descent, and the final one a "method of prevailing deviations."

Both graphical and analytic methods are described, and all in superabundant detail. *A. S. Householder.*

Salzer, Herbert E. Formulas for inverse osculatory interpolation in the complex plane. J. Res. Nat. Bur. Standards **59** (1957), 233–238.

Improved formulas for inverse osculatory interpolation in the complex plane are obtained by inversion of Hermite's formula and the use of appropriate grid point configurations. They cover the cases for $n=2(1)7$, where n is the number of points required in direct osculatory interpolation. The formulas provide an improved means for inverse interpolation in the complex plane where the first derivative is either tabulated alongside the function or is easily obtained. *Author's summary.*

Head, J. W.; and Oulton, G. M. Fitting curves to experimental data by least squares. Aircraft Engrg. **29** (1957), 268–270.

Jacobs, Walter. Loss of accuracy in simplex computations. Naval Res. Logist. Quart. **4** (1957), 89–94.

The problem of round-off propagation in the calculation of discontinuous computational algorithms (such as the simplex algorithm of linear programming) is discussed. The author notes that build-up of propagated error may cause the automatic computer to proceed along an incorrect branch of the algorithm. The simplex algorithm is described in detail, and the three possible "wrong turns" in the decision process discussed. No general theory is worked out. This paper may be considered more an empirical than a theoretical discussion of the possibility of obtaining a wrong basis as the solution of the simplex algorithm due to round-off error giving an incorrect decision somewhere during the course of a program.

John W. Carr, III (Ann Arbor, Mich.).

★**Greenwald, Dakota Ulrich. Linear programming: An explanation of the simplex algorithm.** The Ronald Press Company, New York, N.Y., 1957. vii+75 pp. $3.00.

Exposition of the simplex algorithm as used in hand-computed solutions of linear programming problems.

Altman, M. On the approximate solution of linear algebraic equations. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 365–370, XXIX. (Russian summary)

Given the linear system $Ax=b$ with nonsingular matrix A of order n, and a basis e_i for the space, one can form a sequence of vectors x_ν approximating x by projecting $r_\nu=b-Ax_\nu$ orthogonally upon the vectors Ae_i in cyclic order. The method was described by the reviewer [Principles of numerical analysis, McGraw-Hill, New York, 1953; MR **15**, 470] with a geometric sketch of a proof. The author supplies an analytic proof and points out that with suitable choice of metric the method includes the Gauss-Seidel iteration for the system $A^*Ax=A^*b$.

A. S. Householder (Oak Ridge, Tenn.).

Altman, M. An approximation process for the Gaussian least squares principle in the error theory. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 371–374, XXX. (Russian summary)

The conclusions of the paper reviewed above are applied to the solution of the least-square problem $Ax=b+v$, $v^Tv=$ min. *A. S. Householder.*

Kogbetliantz, E. G. Solution of linear equations by diagonalization of coefficients matrix. Quart. Appl. Math. **13** (1955), 123–132.

Um die Verdoppelung der Ordnung bei Aufspaltung eines komplexen linearen Gleichungssystems in zwei reelle zu vermeiden, wird Diagonalisierung der komplexen Matrix $A=A_0$ durch zwei unitäre Matrizen U, T: $^*U'AT=K$ nach folgendem Iterationsverfahren vorgeschlagen: $A_n={}^*u_n'(\zeta_n)A_{n-1}u_n(z_n)$, $K=\lim_{n\to\infty}A_n$, d.h. $A_n={}^*U_n'AT_n$, $T_n=\prod_1^n u_k(z_k)$, $U_n=\prod_1^n u_k(\zeta_k)$, $T=\lim_{n\to\infty}T_n$, $U=\lim_{n\to\infty}U_n$. Die u_n sind sehr einfache unitäre Matrizen, die als Drehungen um den komplexen Winkel z_n bzw. ζ_n gedeutet werden können. In einer festgelegten Reihenfolge wird bei jedem Iterationsschritt ein außerhalb der Diagonale stehendes Element und sein Spiegelbild zu Null gemacht, während die Quadratsumme der Diagonalelemente ständig wächst. Nach Durchlaufen eines Zyklus von $N=m(m-1)/2$ Schritten (m Ordnung von A) ist jede Stelle außerhalb der Diagonale einmal annuliert worden, der Betrag der Nichtdiagonalelemente von A_N ist dabei durchschnittlich um den Faktor e^{-1} keiner als bei A_0. Es werden detaillierte Rechenvorschriften (nebst Vereinfachungen für symmetrische komplexe, allgemeine reelle, Hermitesche und schief-Hermitesche Matrizen) gegeben.

J. Weissinger (Zbl **66** (1957), 101).

★**Sinden, Frank W. An oscillation theorem for algebraic eigenvalue problems and its applications.** Thesis, The Swiss Federal Institute of Technology, Zurich, 1954. 59 pp.

Es sei $V(X)$ die Zahl der Zeichenwechsel in der reellen Folge $X=(x_1, \cdots, x_n)$, wobei etwaige Nullen unter den x_p außer acht zu lassen sind. $AX=Y$ sei eine lineare Transformation; dabei heißt die Matrix $A=(a_{jk})$ variationsvermindernd (bzw. -vermehrend), wenn für jedes X gilt $V(Y)\leqq V(X)$, (bzw. $V(Y)\geqq V(X)$). Z.B. ist der Übergang von der Folge $x_1, \cdots, x_n$ zu $x_1, (x_1+x_2)/2, (x_2+x_3)/2, \cdots, (x_{n-1}+x_n)/2, x_n$ variationsvermindernd. Beim Übergang zur Differenzenreihe $D=(x_2-x_1, \cdots, x_n-x_{n-1})$ gilt $V(D)\geqq V(X)-1$, ferner im Falle $x_1=x_n=0$, $X\not\equiv 0$ sogar $V(D)\geqq V(X)+1$, und schließlich $V(D)\geqq V(X)$, falls x_1 oder x_n gleich null ist. In der Eigenwertaufgabe $AX=\lambda DX$ sei A eine n-reihige symmetrische positiv definit variationsvermindernde (bzw. -vermehrende) Matrix mit $a_{j,j+1}\neq 0$, $a_{j+1,j}\neq 0$ (für $j=1, \cdots, n-1$) und D eine Diagonalmatrix mit positiven Elementen. Dann gelten die den klassischen Sturm-Liouvilleschen Sätzen analogen Aussagen: Alle Eigenwerte λ_j sind positiv und einfach, $\lambda_1>\lambda_2>\cdots>\lambda_n>0$. Ein zu λ_k gehöriger Eigenvektor hat genau $k-1$ (bzw. $n-k$ bei variationsvermehrender Matrix) Zeichenwechsel. Für einen Eigenvektor $U=(u_1, \cdots, u_n)$ ist $u_1\neq 0$, $u_n\neq 0$, $u_p=0$ nur dann, wenn $u_{p-1}u_{p+1}<0$ ist. Verf. wendet diese Sätze an auf den transversal schwingenden, inhomogenen Stab bei verschiedenen Arten der Lagerung, insbesondere auch auf

den mehrfach gestützten und den mehrfach gelenkig gegliederten Stab. Allgemein führt für einen Differentialoperator gerader Ordnung

$$(d/dx)\{p_{\mu-1}(d/dx)\{\cdots\{p_1 d/dx\{\cdots\}\} \text{ mit } p_k(x)=p_{\mu-k}(x)$$

bei gewissen weiteren Voraussetzungen (über die hinzukommenden Randbedingungen usw.) der zugehörige Differenzenoperator zu einer variationsvermehrenden Matrix. Bei Benutzung eines Einschließungssatzes kann man durch Abzählung der Zeichenwechsel jeweils die Nummer der Differenzeneigenwerte festlegen.

L. Collatz (Zbl **55** (1955), 352).

★ **Peltier, Jean. Résolution numérique des équations algébriques.** Gauthier-Villars, Paris, 1957. iv+244 pp. 2500 francs.

Cet ouvrage se propose de fournir aux non-spécialistes de l'Analyse Numérique une méthode pratique de résolution des équations algébriques. L'auteur ne cherche donc pas à dresser un tableau d'ensemble des divers procédés de résolution connus, avec leurs avantages respectifs. Au contraire, il développe une méthode particulière (celle de Graeffe) et il étudie avec beaucoup de détails les difficultés qui peuvent intervenir dans la pratique. Pour la recherche des arguments des racines, il recommande de faire le changement d'inconnue: $x=r(1+iZ)/(1-iZ)$, r étant le module précédemment déterminé. Si en outre les valeurs obtenues pour les racines sont insuffisamment précises, il propose de les améliorer par la méthode de Newton.

L'ouvrage est élémentaire; la théorie est développée sommairement, sans recherche particulière de rigueur. L'auteur insiste surtout sur l'aspect pratique du problème et il fait de très nombreuses remarques, souvent excellentes, sur les causes d'erreurs ou de difficultés, en les illustrant d'exemples numériques traités complètement.

Bien que résolvant surtout des équations à coefficients réels, l'auteur consacre une dizaine de pages aux équations à coefficients complexes. D'autre part, l'ensemble de l'ouvrage (malgré une brève allusion, p. 164, au calcul électronique) ne s'occupe que du calcul à la main sur machines de bureau.

Quelques erreurs ou omissions relevées à la lecture: p. 81, la référence omise est: "Carvallo, 2ème Thèse, Paris, 1890"; p. 83 et p. 161, la détermination de l'argument d'une racine par sa tangente est insuffisante; et p. 215, le tableau II bis est inexact et ne correspond pas aux résultats du tableau II. *G. Brillouet* (Nantes).

Markovitch, Drag. Les méthodes pratiques de factorisation approximative des polynomes. Bull. Soc. Math. Phys. Macédoine **7** (1956), 5–16. (Serbo-Croatian summary)

In two preceding articles [Bull. Soc. Math. Phys. Serbie **4** (1954), 3–11; **8** (1956), 53–58; MR **17**, 88; **18**, 787], the author states he has given a method of approximate factorization of polynomials. By this is meant the following: The polynomial must be represented as

$$\sum_{v=0}^{m} x^v a_v=\Big(\sum_{v=0}^{k} x^v p_v(n)\Big)\cdot\Big(\sum_{v=0}^{m-k} x^v q_v(n)\Big)+\sum_{v=0}^{h} x^v \gamma_v(n)$$

$$(k=1, 2, \cdots, m-1;\ h\leq m)$$

such that, when $n\to\infty$, the sequences $p_v(n)\to P_v\neq 0$, $q_v(n)\to Q_v\neq 0$, $\gamma_v(n)\to 0$. The sequences of polynomials $\sum_{v=0}^{m} x^v p_v(n)$ and $\sum_{v=0}^{m-k} x^v q_v(n)$ represent sequences of approximate factors of the given polynomial $\sum_{v=0}^{m} x^v a_v$. The results applied here concern the case $k=1$. In this paper are treated the cases which generalize the method and show the efficiency of the method in practice. The principal result is as follows: Since x of the algebraic equation (*) $\sum_{v=0}^{m} x^v a_v=0$ has been represented as a rational function $x=\sum_{v=0}^{m-1} x^v a_{1v}/\sum_{v=0}^{m-1} x^v a_{0v}$, it is necessary by the indicated process of iteration to form the initial matrix A and the generating matrix A^n, $A=\|a_{ij}\|$, $A^n=\|a_{ij}(n)\|$ $(i, j=0, 1, 2, \cdots, m-1)$. Then, if the generating matrix A^n tends, for $n\to\infty$, toward a matrix M in such a way that the ratio of two elements which correspond to the same column of two consecutive lines, tends toward $\lim_{n\to\infty} a_{ij}(n)/a_{i-1,j}(n)=\lambda\neq 0$ $(i=1, 2, \cdots, m-1,$ $j=0, 1, \cdots, m-1)$, λ represents a real zero of equation (*). A number of numerical examples are given.

E. Frank (Chicago, Ill.).

Stein, P.; and Peck, J. E. L. On the numerical solution of Poisson's equation over a rectangle. Pacific J. Math. **5** (1955), 999–1011.

The authors approximate $z_{xx}+z_{yy}=f(x, y)$ in a rectangular region by a 5-point difference equation with $\rho=\Delta y/\Delta x$ not necessarily equal to 1. By careful matrix analysis they obtain a formula yielding the difference solution along any column of mesh points. However, following Hyman [Appl. Sci. Res. B. **2** (1952), 325–351; MR **13**, 993], they suggest using their formula to calculate selected columns, obtaining the remaining columns stepwise (there is a 3-term recursion relation among the columns). The author's formulas are simplest when $\rho=1$, and they give a perturbation procedure for calculating the difference solution when ρ is near 1 (as may be convenient in practice) and prove conditions for its convergence. They suggest preparation of certain not-very-elaborate tables for $\rho=1$ which would facilitate solution and probably assist in an iterative attack on non-rectangular regions. *M. A. Hyman* (Ossining, N.Y.).

Wasow, Wolfgang R. The accuracy of difference approximations to plane Dirichlet problems with piecewise analytic boundary values. Quart. Appl. Math. **15** (1957), 53–63.

The truncation error of the five-point mesh formula for the approximate solution of the Dirichlet problem in the plane is studied by the method of discrete potential for regions whose boundaries are simple closed analytic curves with boundary values which are continuous and piecewise analytic. It is shown that in closed subregions of a given region the error is uniformly of the order of the mesh width. A method for the solution of the five-point discrete analogue of the Dirichlet problem for the Laplace equation based upon the discrete Green's function is given which is similar to a method proposed by the reviewer [NACA Tech. Note no. **4086** (1958)]. *C. Saltzer.*

Kotal, Miroslav. Relaxationsmethode. Österreich. Ing.-Arch. **11** (1957), 93–102.

Difference equations to be used for the approximate solution of elliptic partial differential equations are derived by Taylor expansions together with the usual error estimates. The solution by relaxation methods is considered and the significance of the residues is discussed. Boundary values are treated by linear and quadratic interpolation and a modification of the relaxation method when a singularity is present is given.

C. Saltzer (Cleveland, Ohio).

Garabedian, P. R. Estimation of the relaxation factor for small mesh size. Math. Tables Aids Comput. **10** (1956), 183–185.

The author examines the well-known method of successive over-relaxation for Laplace's difference equation over a two-dimensional region D. Regarding successive iterations as "time-steps" he shows that the process is analogous to the solution of a hyperbolic difference equation. This enables him to calculate the optional over-relaxation factor r and, using an inequality from eigenvalue theory, obtain a good approximation to r. As a by-product, it becomes clear why it is better to overestimate r than under-estimate it and why optimal r is slightly smaller for the 9-point than for the 5-point difference approximation to the Laplacian. [For a related treatment, see Hyman, Bull. Amer. Math. Soc. **63** (1957), 131, 132.] *M. A. Hyman* (Philadelphia, Pa.).

Elrod, H. G., Jr. New finite-difference technique for solution of the heat-conduction equation, especially near surfaces with convective heat transfer. Trans. A.S.M.E. **79** (1957), 1519–1525, discussion 1525–1526.

Juncosa, M. L.; and Young, David. On the Crank-Nicolson procedure for solving parabolic partial differential equations. Proc. Cambridge Philos. Soc. **53** (1957), 448–461.

Consider the diffusion equation (*) $u_t=u_{xx}$ for $0\leqq x\leqq 1$ and $t>0$, with boundary condition $u(0+, t)=u(1-, t)=0$ ($\geqq 0$) and initial condition $u(x, 0+)=f(x)$ $(0<x<1)$. J. Crank and P. Nicolson [Proc. Cambridge Philos. Soc. **43** (1947), 50–67; MR **8**, 409] proposed replacing (*) by the stable difference equations

$$(\Delta t)^{-1}[U(x, t+\Delta t)-U(x, t)]=$$
$$\tfrac{1}{2}[\delta_x^2U(x, t)+\delta_x^2U(x, t+\Delta t)],$$

where

$$\delta_x^2U(x, t)=(\Delta x)^{-2}[U(x+\Delta x, t)-2U(x, t)+U(x-\Delta x, t)],$$

but gave no proof of convergence, as Δt, $\Delta x\to 0$. The present authors supply convergence proofs under suitable hypotheses, and study this order of convergence in detail for various classes of f. They always take $\Delta x=M^{-1}$ $(M=1, 2, \cdots)$ and $\Delta t<M^{-1}\{t_0/3\log M\}^{\frac{1}{2}}$, and the theorems hold in $R=\{(x, t)|t\geqq t_0>0$ and $0\leqq x\leqq 1\}$.

Convergence theorem: If f is of bounded variation and continuous for $0\leqq x\leqq 1$ except for a denumerable set of jumps at which $f(x)=2^{-1}[f(x+)+f(x-)]$, then $U_M(x, t)\to u(x, t)$ uniformly in R, as $M\to\infty$.

Let $E(M, t_0)=\sup|U_M-u|$, where the sup is taken over all mesh points in R. Typical order of convergence theorems: (1) If $f(x)$ is 1 for $\alpha\leqq x\leqq\beta$ and 0 elsewhere, then $E(M, t_0)=O(M^{-1})$ as $M\to\infty$, in general. However, if $M\to\infty$ so that each net is a refinement of its predecessor, and if $(\alpha, 0)$, $(\beta, 0)$ are eventually mesh points, then $E(M, t_0)=O(M^{-2})$. (2) If $f(x)=ax+b$ for $0\leqq x\leqq 1$, then $E(M, t_0)=O(M^{-2})$. (3) If f satisfies a Lipschitz condition of order α $(0<\alpha\leqq 1)$, then $E(M, t_0)=O(M^{-\alpha})$.

{Apparent misprint: In the statements of Lemma 4.3, Theorem 4.4, Theorem 6.1, Corollary 6.2, instead of (3.7) read (3.1).} *G. E. Forsythe* (Stanford, Calif.).

Douglas, Jim, Jr. On the errors in analogue solutions of heat conduction problems. Quart. Appl. Math. **14** (1956), 333–335.

Let R denote a bounded region in the plane, with boundary S. The boundary value problem $u_{xx}+u_{yy}=u_t$ [(x, y) in R, $0<t\leqq T$], $u(x, y, 0)=f(x, y)$, $u=g(x, y, t)$ when (x, y) is on S, is approximated by a system of ordinary differential equations of first order with respect to t, by replacing derivatives with respect to x and y by finite differences. Under hypotheses stated in his proof the author shows that the differenece between the solution $w_{ij}(t)$ of the approximating problem and the solution $u(x, y, t)$ of the given problem is of the order of the second power, $(\Delta x)^2$ or $(\Delta y)^2$, of the differences employed.

R. V. Churchill (Ann Arbor, Mich.).

Fujita, Hiroshi. On the error of the finite difference method in 1-dimensional boundary value problems. J. Phys. Soc. Japan **11** (1956), 160–169.

The author studies, using functional analysis, the truncation error when solving boundary-value problems in one dimension using finite difference methods. In particular he considers the problem

$$-y''+qy=f,\ y(0)=y(1)=0\ (q\geqq 0)$$

when it is approximated by a difference problem (whose solution is u) in the usual way. Starting from a theorem of Kato, the author obtains bounds for $R=\max|y-u|$ in terms of q, f, and u, as well as some bounds which involve only q, f. The paper concludes by studying the degree of convergence of the difference solution; that is, it is established that $R=O(h^\alpha)$, and α is determined for various continuity restrictions on q, f; h is the interval between points in the difference solution.

M. A. Hyman (Philadelphia, Pa.).

Sibuya, Masaaki; and Toda, Hideo. Tables of the probability density function of range in normal samples. Ann. Inst. Statist. Math., Tokyo **8** (1957), 155–165.

Using a method due to Cadwell [Ann. Math. Statist. **24** (1953), 603–613; MR **15**, 452] the authors give tables of the probability density function of the range w in normal samples of sizes 3(1)20 for $w=0(0.05)7.65$.

D. M. Sandelius (Göteborg).

See also: Computing Machines: Harmuth. **Structure of Matter:** Belov and Tarhova. **Programming, Resource Allocation, Games:** Symonds; Bellman; Hildreth; Beale; Frank and Wolfe; Jewell; Gale; Tucker; Goldman; Thompson.

Computing Machines

Trefftz, Eleonore. Die Göttinger Rechenmaschinen vom Benutzer aus gesehen. Z. Angew. Math. Mech. **37** (1957), 146–148.

An account of the order-code, of the work organization and of some of the problems solved on the digital computers at the University of Göttingen. *W. F. Freiberger.*

★ **Schecher, Heinz. Programmierung für eine Maschine mit erweitertem Adressenrechenwerk.** Aktuelle Probleme der Rechentechnik. Bericht über das Internationale Mathematiker-Kolloquium Dresden, 22. bis 27. November 1955, pp. 69–81. VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.

One of the most advanced logical structures for any present-day digital computer is that described here for the PERM, the machine at Münich. The author describes a method of incorporating the ordinary separated "index registers" of previous computers into the main memory,

by using several different kinds of indexing plus the concept of "indirect addressing" or automatic address substitution. This system allows automatic storage assignment of subroutines, automatic connection of subroutines in superimposed hierarchy, and automatic iteration recursion (loops within loops), all with greatly increased operational efficiency.

The general logical structure for a computer using such techniques is described. The method of indirect addressing was apparently developed independently in this country for non-hardware (interpretive) procedures by A. Newell and J. C. Shaw in Proceedings of the Western Joint Computer Conference, Los Angeles, 1957.

J. W. Carr, III (Ann Arbor, Mich.).

Tasman, P. Literary data processing. IBM J. Res. Develop. **1** (1957), 249–256.

The main part of this paper presents an account of the adaptation of conventional punched-card methods to the compilation of analytical indexes and concordances of printed works in general and of the Summa Theologica of St. Thomas Aquinas in particular. The scholar still has to examine the entries and group them according to his specific purposes; the mechanical operations are confined to the breaking up of phrases into words, the indexing of these words and the sorting and collating of the various card types. Nevertheless, the indexing of the complete work of St. Thomas Aquinas, estimated to take, manually, 50 scholars 40 years, is reduced, by the help of small machines, to 10 scholars working 4 years. Through the use of large-scale data-processing equipment, this time could be reduced to a quarter, or less than one year.

In a postscript, the author mentions briefly the indexing of the Dead Sea Scrolls programmed for the I.B.M. 705. Here, the program provided for an analysis of the text, taking into account the frequency, use and sequence of words in a particular context. Thus, partially obliterated phrases could be completed by the program.

W. F. Freiberger (Providence, R.I.).

Elliott, D. A. Representation of nonlinear functions of two input variables on analog equipment. Trans. A.S.M.E. **79** (1957), 489–494, discussion 494–495.

This paper discusses graphical methods of realising approximations (valid at least in the working region) of the general function of two variables $z=\phi(x, y)$. The approximations considered are the following: (i) Translations of the same curve $\phi(x, 0)$, viz. $z=\phi[x+f_1(y), 0]+f_2(y)$; (ii) rotations of the same curve, viz. $z=f_3(y)\phi(x, 0)$; (iii) Combinations of (i) and (ii). Such degenerate forms of the general function of two variables are achievable by combinations of analogue function units for functions of one variable.

J. G. L. Michel (Teddington).

Lipton, Stephen. Some statistical applications of electronic computers. Appl. Statist. **6** (1957), 102–113.

A description in general terms of some of the work that has been carried out by the Statistics Department at Rothamsted Experimental Station, using an electronic computer. Particular attention is paid to the routines which have been evolved in connection with the computational work.

From the author's summary.

Harmuth, Henning F. On the solution of the Schroedinger and the Klein-Gordon equations by digital computers. J. Math. Phys. **36** (1957), 269–278.

This paper discusses the numerical solution of the two equations mentioned in the title and contains nothing essentially new.

R. W. Hamming (Murray Hill, N.J.).

Sternlicht, B.; and Maginniss, F. J. Application of digital computers to bearing design. Trans. A. S. M. E. **79** (1957), 1483–1491, discussion 1492–1493.

Lester, Joseph T., Jr. Two applications of a digital computer to machine-design problems. Trans. A.S.M.E **79** (1957), 1476–1482.

Eremeev, N. V. A mechanical device for transforming plane curves. Moskov. Gos. Univ. Uč. Zap. 181. Mat. **8** (1956), 223–233. (Russian)

See also: Numerical Methods: Modern computing methods; Jacobs. **Optics, Electromagnetic Theory, Circuits:** Haselgrove.

PROBABILITY

Getoor, R. K. On characteristic functions of Banach space valued random variables. Pacific J. Math. **7** (1957), 885–896.

The one-one correspondence between positive measures of total measure 1 on a Euclidean E_n and continuous positive-definite functions with $\varphi(0)=1$ on its dual $\hat{E}_n$ can be extended in different versions to a topological vector space V and a suitable dual $\hat{V}$. The author discusses some details for a version formulated by I. Segal [Amer. J. Math. **76** (1954), 721–732; MR **16**, 149], and it is sufficiently "weak" so that by-and-large the measures and the functions become limits of such objects on finite-dimensional subspaces which "exhaust" the vector space and its dual respectively.

S. Bochner (Princeton, N.J.).

Lavendhomme, René. Calcul des probabilités dans certains espaces topologiques. I, II. Ann. Soc. Sci. Bruxelles. Ser. I. **70** (1956), 9–36, 124–163.

Der Verf. versucht, analog zum Verfahren von Riesz-Bourbaki eine Theorie von Maßen in topologischen Räumen, die nicht notwendig lokal kompakt sind, zu entwickeln und eine entsprechende Wahrscheinlichkeitsrechnung zu begründen. Leider ist schon der Beweis des ersten, grundlegenden Lemmas falsch, da die dort betrachtete Funktionenfolge nicht, wie behauptet, gleichmäßig zu konvergieren braucht. Ein ähnlicher Fehler kommt im Beweis des zentralen Grenzwertsatzes für zufällige Variable mit Werten in einem beliebigen Banachschen Raum (Théorème X) vor, und in der Tat gilt der Satz nicht so allgemein (zum Fall eines separablen Hilbertschen Raums vergleiche E. Mourier, Ann. Inst. H. Poincaré **13** (1953), 161–244; MR **16**, 268]. Ferner wird eine Verallgemeinerung des Kolmogorovschen charakteristischen Funktionals definiert. Zahlreiche Ungenauigkeiten und Druckfehler, auch im Literaturverzeichnis und den Hinweisen darauf, erschweren die Lektüre.

K. Krickeberg (Würzburg).

Kampé de Fériet, Joseph. Une classe de mesures de probabilité sur les espaces l^p et L^p ($p \geq 1$). C. R. Acad. Sci. Paris **244** (1957), 1119–1122.

An L-measure on a Banach space $\mathfrak{X}$ is a measure space $(\mathfrak{X}, \mathfrak{B}, \mu)$ defined on the set $\mathfrak{X}$ in such a manner that every bounded linear functional becomes a measurable function. The author wishes to exhibit the existence of L-measures in the spaces L_p ($p \geq 1$) of p-power integrable real functions on some measure space. In order to do so, he first quotes a result of Mazur to the effect that all separable L_p spaces are homeomorphic and shows that the image of an L-measure under this homeomorphism is itself an L-measure [Studia Math. **1** (1929), 83–85]. This leaves him with the task of exhibiting an L-measure in one such L_p space and for this purpose he chooses l_2, the (Hilbert) space of square-summable sequences, and produces a measure having the desired properties. *A. Beck* (New Orleans, La.).

Kampé de Fériet, Joseph. Mesures de probabilité sur un espace de Hilbert séparable. C. R. Acad. Sci. Paris **244** (1957), 1850–1853.

Let R^∞ denote the set of all sequences of real numbers and let l_2 denote the set of those which are square-summable. Let $x \in R^\infty$ and denote by ξ_n that function which assigns to x the number which is its nth coordinate. In the paper reviewed above, the author exhibits a class of measures on l_2 with the following properties: (a) $\mu(l_2)=1$, (b) each ξ_n is measurable, (c) $\sum_{n=1}^{\infty} \int_\Omega \xi_n^2 d\mu < \infty$. In this note, he constructs all the measures with these properties. Let (Ω, ζ, μ) be any measure space and let $\{\xi_n\}$ be any sequence of square-integrable real functions defined on it for which $\sum_{n=1}^{\infty} \int_\Omega \xi_n^2 d\mu < \infty$. Then for each $\omega \in \Omega$, define $T(\omega)$ as the sequence $\{\xi_n(\omega)\}$. This is shown to be square-summable for almost every $\omega \in \Omega$, and thus defines in l_2 a measure of the desired type. The author then uses an extension of a result of Fréchet to produce the following analog of the strong law of large numbers. If $\mathfrak{X}$ is a separable real Hilbert space and $x_1, \cdots, x_n, \cdots$ a sequence of square-summable random variables, each corresponding to an L-measure in $\mathfrak{X}$ and each having expectation 0, then if $\sum_{n=1}^{\infty} E(\|x_n\|^2) < \infty$, the series $x_1 + \cdots + x_n + \cdots$ converges strongly with probability 1. *A. Beck* (New Orleans, La.).

Kampé de Fériet, Joseph. Mesures de probabilité sur les espaces de Banach admettant une base dénombrable. C. R. Acad. Sci. Paris **244** (1957), 2450–2454.

Utilisant la base de Schauder [Math. Z. **26** (1927), 47–65], l'auteur détermine des mesures de probabilité sur l'espace $C=C(0, 1)$ possédant la propriété suivante: un élément de C est presque sûrement une fonction à variation bornée. La mesure de probabilité considérée par Wiener dans ses recherches sur le mouvement brownien [Acta Math. **55** (1930), 117–258] ne possède pas cette propriété. *J. Deny* (Strasbourg).

Kampé de Fériet, Joseph. Mesures de probabilité sur l'espace de Banach C [0,1]. C. R. Acad. Sci. Paris **245** (1957), 813–816.

Modifiant légèrement la construction de la note précédente [voir analyse ci-dessus], l'auteur obtient une classe plus générale de mesures de probabilité sur C, contenant cette fois-ci la mesure de Wiener. *J. Deny* (Strasbourg).

Ryll-Nardzewski, C. On stationary sequences of random variables and the de Finetti's equivalence. Colloq. Math. **4** (1957), 149–156.

This paper gives a new treatment of a theory developed in various stages of generality by de Finetti, Hencin, Dynkin, and Hewitt and Savage [for bibliography see Hewitt and Savage, Trans. Amer. Math. Soc. **80** (1955), 470–501; MR **17**, 863, q. v. for terminology and notation]. The author proves that every countably additive symmetric probability on $\tilde{\mathfrak{X}}$ is representable as a certain integral of product probabilities if X is the real line and $\mathfrak{X}$ is the σ-algebra of Borel subsets of X. The integral is not, as in all previous treatments, taken over the space $\mathbf{P}$ of probabilities on $\mathfrak{X}$. Instead, the author proceeds as follows. For $a=(a_1, a_2, a_3, \cdots, a_n, \cdots) \in \tilde{X}$, let $\phi(a)=(a_2, a_3, \cdots a_n, \cdots)$. Let $\tilde{\mathfrak{X}}_{\text{inv}}$ be the family of all sets $E \in \tilde{\mathfrak{X}}$ such that $\phi^{-1}E=E$. Let μ be a symmetric (countably additive) probability on X. Let τ be μ restricted to $\tilde{\mathfrak{X}}_{\text{inv}}$, and $\mu(E|x)$ the $\tilde{\mathfrak{X}}_{\text{inv}}$-measurable function on $\tilde{X}$ such that $\mu(E \cap Q)=\int_Q (E|x)d\mu(x)$ for all $Q \in \tilde{\mathfrak{X}}_{\text{inv}}$ and $E \in \tilde{\mathfrak{X}}$. Then $\mu(\cdot|x)$ is a product measure almost everywhere (μ) and $\mu(E)=\int_X \mu(E|x)d\tau(x)$ for all $E \in \tilde{\mathfrak{X}}$. This is a reformulation of a special case of Hewitt and Savage loc. cit., Theorem 7.4. The author's Theorem 6 (γ) is Theorem 3.2 of Hewitt and Savage loc. cit. *E. Hewitt* (Seattle, Wash.).

Castoldi, Luigi. Attorno a un problema probabilistico di occupazione. Atti Accad. Ligure **11** (1954), 119–126 (1955).

Each of r balls is deposited in a cell selected at random out of the n given cells, the probability of being selected being equal to $1/n$ for each cell and the successive selections being mutually independent. Let $k_1 < k_2 < \cdots < k_s$ be arbitrary fixed non-negative integers. For $m=1, 2, \cdots, s$, the symbol $X_m(r, n)$ denotes a random variable representing the number of those cells that contain exactly k_m balls. Generalizing the result of R. v. Mises [Rev. Fac. Sci. Univ. Istanbul (N.S.), **4** (1939), 145–163], the author shows that, if n and $r=n\rho$ are indefinitely increased in such a manner that the quantities $ne^{-\rho}\rho^{k_m}$ remain bounded, then the joint distribution of the variables $X_m(r, n)$ is arbitrarily approximated by that of s mutually independent Poisson variables.
J. Neyman (Berkeley, Calif.).

Rutherford, R. S. G. On a contagious distribution. Ann. Math. Statistics **25** (1954), 703–713.

Verf. behandelt den Spezialfall der Woodburyschen Verteilung [Ann. Math. Statist. **20** (1949), 311–313; MR **10**, 720] $p_x=a+bx$=Wahrsch. für Erfolg nach x bisherigen erfolgreichen Urnenziehungen, und behandelt das Problem der Anpassung an gegebene Werteverteilungen und die Approximation durch die negative binomische Verteilung und nach Gram-Charlier. Der Nutzen dieses Verteilungsschemas wird an zwei empirischen Zahlenreihen, die Güte der Annäherungen in einer Tabelle aufgezeigt. *D. Morgenstern* (Zbl 56 (1955), 359).

Barra, Jean-René. Sur une propriété des fonctions de répartition empiriques considérées comme "estimations" des fonctions de répartition théoriques. C. R. Acad. Sci. Paris **244** (1957), 3020–3022.

Let the empiric distribution function $F_n^*(z)$ estimate the exact distribution $F(z)$ in the following sense: (1) $F_n^*(z)$ converges almost completely to $F(z)$ for all z; (2) $F_n^*(z)=F(z)+\delta_n(z)/n^{\frac{1}{2}}$ implies that $\delta_n(z)$ converges in law to a semi-reduced Laplacian random function of covariance $F(t, \tau)=F(t)-F(t)F(\tau)$ where $t \leq \tau$. The author shows that, given a measurable function $Z=$

$\Phi(X^1, \cdots, X^p)$ of p independent random variables, if one takes for the law of X^i the empiric law deduced from known observations, then the empiric law obtained for Z estimates in the preceding sense the exact law for Z.

H. P. Edmundson (Santa Monica, Calif.).

Ibragimov, I. A. A theorem in the theory of infinitely divisible laws. Teor. Veroyatnost. i Primenen. **1** (1956), 485–489. (Russian. English summary)

Let $\mathfrak{F}$ be a class of infinitely divisible distribution functions F for which, if $F \in \mathfrak{F}$ and $F*H=Q$ where $Q(x)$ is an infinitely divisible distribution function, it follows that H is also an infinitely divisible distribution function. The following theorem is proved. Class $\mathfrak{F}$ is identical to the set of all normal distributions.

Author's summary.

Kolmogorov, A. N. Deux théorèmes asymptotiques uniformes pour des sommes des variables aléatoires. Teor. Veroyatnost. i Primenen. **1** (1956), 426–436. (Russian. French summary)

Soit $\xi_1, \cdots, \xi_n$ des variables aléatoires indépendantes, $\xi=\xi_1+\cdots+\xi_n$ et $\Phi, F_1, \cdots, F_n$ les fonctions de distribution correspondantes. Désignons par $\mathfrak{E}$ l'ensemble des distributions dégénérées $E(x)=0$ pour $x \leqq a$ et 1 pour $x>a$, et par Θ l'ensemble des distributions infiniment divisibles.

Théorème I. Il existe une constante C telle que pour tout $\varepsilon>0$, $L>2l>0$ les inégalités $E_k(x-l)-\varepsilon \leqq F_k(x) \leqq E_k(x+l)+\varepsilon$ $(E_k \in \mathfrak{E}; k=1, \cdots, n; -\infty<x<+\infty)$ impliquent l'existence d'une fonction $\Psi \in \Theta$ pour lequel on a $\Psi(x-L)-\delta \leqq \Phi(x) \leqq \Psi(x+L)+\delta (-\infty<x<+\infty)$ avec

$$\delta=C\max\{Ll^{-1}(\log Ll^{-1})^{\frac{1}{2}}, \varepsilon^{1/4}\}.$$

Dans le cas des distributions F_k égales $F_k(x)=F(x)$ $(k=1, \cdots, n)$ on démontre un théorème plus simple.

Théorème II. Il existe une constante C telle que pour tout n et F on peut trouver une fonction $\Psi \in \Theta$ verifiant l'inégalité $|\Psi(x)-\Phi(x)| \leqq Cn^{-1/4}$ $(-\infty<x<+\infty)$.

Résumé de l'auteur.

Kac, M. Uniform distribution on a sphere. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 485–486, XL. (Russian summary)

The author proves the following theorem, using characteristic functions: If a point moves continuously on a sphere in such a manner that its projection on any diameter covers it uniformly in unit time, then the point covers the sphere uniformly.

E. Lukacs.

Hanš, Otto. A note on negative binomial distribution. Apl. Mat. **2** (1957), 222–226. (Czech. Russian and English summaries)

In this paper there are given two formulae for "tails" of negative binomial distribution, an approximate formula:

$$\sum_{k=n}^{\infty} \binom{\beta+k-1}{k} p^{\beta} q^k = \sum_{j=0}^{r-1} \binom{n+\varepsilon+j-1}{n-1} p^{j+\varepsilon} q^n + Z,$$

where the remainder Z satisfies the inequality

$$0 \leqq Z \leqq \varepsilon(\varepsilon+1)q^n/p^{1-\varepsilon},$$

n and r are natural numbers, $\beta=r+\varepsilon$, $0 \leqq \varepsilon \leqq 1$, and an exact formula:

$$\sum_{j=0}^{c} \binom{\beta+j-1}{j} p^{\beta} q^j = \sum_{j=0}^{c} \binom{\beta+c}{j} p^{\beta+c-j} q^j.$$

Author's summary.

★ **Kendall, D. G.; and Reuter, G. E. H. Some pathological Markov processes with a denumerable infinity of states and the associated semigroups of operators on l.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 377–415. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Let $P(t)=(p_{ij}(t))$, $t \geqq 0$, be quasi-stochastic matrices: $p_{ij}(t) \geqq 0$, $\sum_{j=1}^{\infty} p_{ij}(t)=1$, $p_{ij}(u+v)=\sum_{\alpha=1}^{\infty} p_{i\alpha}(u)p_{\alpha j}(v)$, $p_{ij}(0)=\delta_{ij}$ and $\lim_{t \downarrow 0} p_{ii}(t)=1$. It is known [A. Kolmogoroff, Moskov. Gos. Univ. Uč. Zap. **148** Mat. **4** (1951), 53–59; MR **14**, 295; J. L. Doob, Trans. Amer. Math. Soc. **58** (1945), 455–473; MR **7**, 210; D. G. Kendall, ibid. **78** (1955), 529–540; MR **16**, 725] that the limits $q_{ij}=\lim_{t \downarrow 0} t^{-1}p_{ij}(t)$ $(i \neq j)$ and $q_i=\lim_{t \downarrow 0} t^{-1}(1-p_{ii}(t))$ exist, q_{ij} is always finite and $\sum_{j \neq i} q_{ij} \leqq q_i \leqq \infty$. The case for which $\sum_{j \neq i} q_{ij}=q_i<\infty$ holds for each i was discussed by Hille [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 20–25; MR **15**, 706], Feller (unpublished) and Kato [J. Math. Soc. Japan **6** (1954), 1–15; MR **15**, 965] with the method of the semi-group $(P_t x)_j=\sum_{i=1}^{\infty} x_i p_{ij}(t)$, defined for x of the Banach space l. Two pathological examples of the quasi-stochastic matrices were given by Kolmogoroff, loc. cit.: (K1) and (K2) respectively corresponding to the cases $q_1=\infty$ and $\sum_{j \neq 1} q_{1j}<q_1<\infty$. The authors discuss these examples from the semi-group theoretic point of view. Let $a_\alpha>0$ $(\alpha \geqq 2)$ and $\sum_\alpha a_\alpha^{-1}<\infty$, and let Ω be defined by $(\Omega x)_1=\sum_\alpha (a_\alpha x_\alpha - x_1)$, $(\Omega x)_\beta=x_1-a_\beta x_\beta$ $(\beta \geqq 2)$ for those $x \in l$ which satisfy $\sum_\alpha |a_\alpha x_\alpha - x_1|<\infty$. It is shown that the semi-group P_t in l with Ω as its infinitesimal generator gives the quasi-stochastic matrix $P(t)$ of the example (K1). Next let $a_1=1$, $a_2=0$, $a_\beta>0$ $(\beta \geqq 3)$ and $\sum_\beta a_\beta^{-1}<\infty$, and let Ω be defined by $(\Omega x)_1=-x_1$, $(\Omega x)_2=a_3x_3$, $(\Omega x)_\beta=a_{\beta+1}x_{\beta+1}-a_\beta x_\beta$ $(\beta \geqq 3)$ for those $x \in l$ for which $\sum_{j=1}^{\infty} |a_{j+1}x_{j+1}-a_jx_j|<\infty$, $\lim_{j \to \infty} a_jx_j=x_1$. Such an Ω is the infinitesimal generator of a semi-group P_t in l which gives the quasi-stochastic matrix $P(t)$ of the example (K2) satisfying the stochastic condition $\sum_{j=1}^{\infty} p_{ij}(t)=1$. The authors also obtain the degenerate forms of the "backward" and "forward" differential equations corresponding to the examples (K1) and (K2). They also discuss the actual stochastic motions of these examples.

K. Yosida (Tokyo).

Kendall, David G. Some further pathological examples in the theory of denumerable Markov processes. Quart. J. Math. Oxford Ser. (2) **7** (1956), 39–56.

Let $\{P_{ij}(t) : t \geqq 0; i, j \in S\}$ be the transition probabilities for a temporally homogeneous Markov process with a denumerable set S of states, and let $q_{ij}=p_{ij}'(+0)=\lim_{t \to +0} t^{-1}(P_{ij}(t)-\delta_{ij})$. The examples discussed in this paper include the following: (1) A set $\{q_{ij}\}$ is exhibited, with q_{ij} finite for all i, j and $\sum_j q_{ij}=0$ for each i, such that there exist continuum many distinct processes $\{p_{ij}(t)\}$, all of them having $p_{ij}'(+0)=q_{ij}$ and satisfying both the backward equations $p_{ij}'(t)=\sum_k q_{ik}p_{kj}(t)$ and the forward equations $p_{ij}'(t)=\sum_k p_{ik}(t)q_{kj}$. The non-unicity for the solutions of both sets of differential equations exhibited here is much more drastic than in previously known examples. (2) A process $\{p_{ij}(t)\}$ is constructed for which one state, labelled 0, has the property that $q_{00}=-\infty$ and $q_{0j}=0$ for $j \neq 0$. This may also be described as more extreme behaviour than occurs in a previous example, due to Kolmogorov [Moskov. Gos. Univ. Uč. Zap. **148**, Mat. **4** (1951), 53–59; MR **14**, 295], which had $q_{00}=-\infty$ but $q_{0j}=1$ for $j \neq 0$.

The processes treated in this paper are all constructed

explicitly by writing down the infinitesimal generator of the semigroup $\{P_t\}$ of transition operators on the Banach space l, defined by $(P_t x)_j = \sum_i x_i p_{ij}(t)$ $(x \in l)$ and using the Hille-Yosida theorem. The Laplace transforms of the p_{ij} can then be found by examining the resolvent operator. Similar methods were used by Kendall and Reuter [see the paper reviewed above] in order to discuss examples due to Kolmogorov [loc. cit.].

G. E. H. Reuter (Manchester).

Parzen, Emanuel. On consistent estimates of the spectrum of a stationary time series. Ann. Math. Statist. 28 (1957), 329–348.

The author studies the estimation of the spectral density, distribution function or other spectral averages for stationary stochastic processes. The process is supposed to be normal or of a more general form, whose fourth order mixed moments do not differ too much in a certain sense from those of a normal process. Both discrete and continuous processes are treated. After studying some statistical properties of the sample covariances, the author considers the estimation of the spectral density. His main results describe the bias and covariances of certain estimates in asymptotic terms, relating them to the infinitesimal properties of the spectrum and the estimate used. Alternative estimates are discussed from this point of view. It is shown that the whole graph of the estimate can be obtained by an interpolation procedure. *U. Grenander* (Providence, R.I.).

John, Peter W. M. Divergent time homogeneous birth and death processes. Ann. Math. Statist. 28 (1957), 514–517.

In a time-homogeneous birth and death process, if at time t the population is n, the probability that a birth (or a death) occurs in the time interval $(t, t+h)$ is $\lambda_n h + o(h)$ (or $\mu_n h + O(h)$, and the probability that more than one event occurs is $o(h)$. Let $n(0)=1$, $\lambda_0=0$, $\mu_1>0$. If $\bar{\tau}_m$ is the expected value (conditional upon non-absorption) of the time taken for a change in population from m to $m+1$, the recursion formula $\bar{\tau}_m = \lambda_m^{-1} + \mu_m \lambda_m^{-1} \bar{\tau}_{m-1}$ is obtained. This leads to a simple derivation of necessary and sufficient conditions for the process to be divergent. Related remarks of interest are made in the paper. *C. Goffman* (Lafayette, Ind.).

Mathews, P. M.; and Srinivasan, S. K. Stochastic processes associated with integrals of a class of random functions. Proc. Nat. Inst. Sci. India. Part A. 22 (1956), 369–376.

The authors calculate the joint characteristic function of the quantities $y_j(t)$ $(j=0, 1, 2, \cdots, m)$, where $y_j(t) = \int_0^t \phi_j(\tau) y_{j-1}(\tau) d\tau$ and $y_{-1}(t)$ is a sequence of delta functions generated from a Poisson process. Moments and frequency functions are calculated for particular cases.

P. Whittle (Wellington).

Fortet, R.; et Canceill, B. Probabilités de perte en selection conjuguée. Teleteknik 1 (1957), 41–55.

The authors first compute some preliminary expectations for some fairly general telephone switching models having the form of Markov processes. The more difficult problems of blocking in switching networks with several "stages" are then considered. It is in the province of the reviewer to point out that in French and Scandinavian telephony the stages denote sets of terminals, while in America a stage is a set of switches; a European $(n+1)$-stage network is an American n-stage network. The author's main topic is then a 3-stage (American 2-stage) network for connecting lines to trunks. Under some simplifying assumptions of independence the probability of loss is approximately evaluated as an integral, and it is suggested by examples that certain much simpler approaches to the loss are quite unacceptable in point of accuracy.

V. E. Beneš (Murray Hill N.J.).

Runnenburg, J. Th. Machines served by a patrolling operator. Math. Centrum Amsterdam. Statist. Afdeling Rep. S 221 (VP 13) (1957), 13 pp.

This paper deals with the following stochastic problem. A repairman is responsible for servicing n identical machines. The machines are numbered modulo n and the repairman inspects each machine in order, repairing any machine that has failed before moving to the next machine. It is assumed that the man requires a fixed to repair a machine are independent random variables, time to travel from one machine to the next that the times and that the life times of a machine before failure are independent random variables with exponential distribution functions.

The author considers the random variable r_m defined as the duration of the complete starting at the moment the operator leaves the mth machine and finishing when the operator leaves the $(m+n)$th machine; and the random variable w_m defined as the time the $(m+n)$th machine has stopped when the operator reaches it. Using some well known properties of Markov chains, the author shows that these random variables have a limit distribution for $m \to \infty$ and that one can evaluate explicitly the asymptotic form of these distributions for $n \gg 1$. The problem for moderate values of n is, however, very complicated. *G. Newell* (Providence, R.I.).

Sprott, D. A. Probability distributions associated with distinct hits on targets. Bull. Math. Biophys. 19 (1957), 163–170.

Suppose that shots are fired at targets, and that any shot has a probability p, not necessarily unity, of striking the target at which it was aimed. Various firing schemes are discussed. The results include: the probability that exactly m targets are hit when s shots are fired on N targets, the probability that exactly k additional targets are hit by u additional shots, the mean and variance of k, the probability distribution of the numbers of shots required to hit exactly m targets. *R. G. Stanton.*

Price, Robert. The detection of signals perturbed by scatter and noise. Trans. I.R.E. PGIT–4 (1954), 163–170.

The following model is considered for analyzing noise-distorted scatter transmissions: There are M possible transmitted signals $x^{(k)}(t) = x_s^{(k)}(t) \sin \omega_0 t + x_c^{(k)}(t) \cos \omega_0 t$, $k=1, 2, \cdots, M$, all Gaussian processes a-priori equally likely. The scattering process is a linear time-invariant functional transformation $x(t) \to z(t)$ such that $\sin \omega_0 t$ goes into the Gaussian process $y_s(t) \sin \omega_0 t + y_c(t) \cos \omega_0 t$, where y_s, y_c are Gaussian, with given identical spectra. The final received signal is $w(t) = z(t) + n(t)$, where $n(t)$ is again Gaussian. Expressions are found for the a-posteriori probabilities $P[x^{(k)}(t)/w(t)]$, $k=1, 2, \cdots, M$, which, as is to be expected, become more tractable in the case of small 'signal-to-noise' ratios. *E. Reich* (Minneapolis, Minn.).

Mycielski, Jan; et Paszkowski, S. Sur un problème du calcul de probabilité. I. Le mouvement d'une molécule sur une droite. Studia Math. **15** (1956), 188–200.

A material endowed with mass is distributed on a line, and a molecule moving on this line may be reflected, transmitted, or absorbed. These phenomena are assumed to be random and independent with respect to non-overlapping segments, but dependent on the mass of the segment and on the direction of motion. The probabilities of transmission, reflection, and absorption for a segment of mass x are designated, respectively, by $p(x)$, $q(x)$, $r(x)$ for a molecule moving from left to right, and by $P(x)$, $Q(x)$, $R(x)$ for a molecule moving from right to left. It is shown that

$$(*)\quad p(x+y)=\frac{p(x)p(y)}{1-Q(x)q(y)},\ q(x+y)=q(x)+\frac{p(x)P(x)q(y)}{1-Q(x)q(y)},$$

and each equation has a counterpart, obtained by interchanging small and capital letters. The authors exhibit the complete solution subject to the requirement that all variables $p, q, \cdots$ lie on [0, 1]. The case $p=P$ had been considered previously by C. Ryll-Nardzewski [Colloq. Math. 4 (1956), 142; presumably unpublished].

{There is an interesting analogy between this random-walk problem and the theory of transmission lines. The structure of (*) is identical with the rule of composition for nonbilateral networks studied in Redheffer, J. Math. Phys. **28** (1950), 237–258 [MR **12**, 147]. Here the case $p\neq P$ is reduced to the case $p=P$ by the change of variable $t=(pP)^{\frac{1}{2}}$. The corresponding functional equations (*) with $p=P$ were introduced and solved for a problem on dielectric media in Redheffer, J. Rational Mech. Anal. **3** (1954), 271–279 [MR **15**, 763]. A generalization is discussed in Redheffer, ibid. **5** (1956), 835–848 [MR **19**, 558]. The latter enables one to solve the probability problem for inhomogeneous media; one gets a Riccati equation $q'=a+2bq+cq^2$ with variable coefficients satisfying $a\geqq 0$, $c\geqq 0$, $a+b\leqq 0$, $b+c\leqq 0$.}

R. M. Redheffer (Göttingen).

See also: Groups and Generalizations: Gourevitch. **Sequences, Series, Summability:** Eyraud. **Integral Transforms:** Devinatz. **Numerical Methods:** Muller. **Elasticity, Plasticity:** Litwiniszyn. **Quantum Mechanics:** Koopman. **Economics, Management Science:** Rosenblatt. **Information and Communication Theory:** Takács; von Sydow.

STATISTICS

★ **Wallis, W. Allen; and Roberts, Harry V. Statistics, a new approach.** The Free Press, Glencoe, Illinois, 1957. 38+646 pp. $6.00.

This is an unusual textbook in statistics. Using only very elementary mathematics the authors introduce most of the basic concepts and ideas of theoretical statistics. Few complete mathematical derivations are given; instead the authors discuss the reason why the statistical techniques work or why they do not. In this way the reader is given explanations and interpretations rather than formal proofs. The discussion is supplemented by a large number of well chosen examples from actual statistical practice. This is one of the most valuable things in this book and should be very helpful to any student of theoretical and applied statistics.

In this way the book covers such important topics as randomness, tests and estimates, confidence intervals, quality control, correlation and regression. The last two chapters contain short-cut methods and some techniques used in time series analysis.

In teaching mathematical statistics this book should be very useful as a complement to the textbooks with a more mathematical orientation. Everyone is likely to learn a good deal from it. *U. Grenander.*

Michetti, Bruno; e Dall'Aglio, Giorgio. La differenza semplice media. Statistica, Bologna **17** (1957), 159–255.

Lucid review of the mean difference: Definition, computation, formal relations and sampling properties. Thorough references. Comprehensive bibliography, including F. R. Helmert, Astr. Nachr. **88** (1876), 113–132.

H. Wold (Uppsala).

★ **Mazzoni, Pacifico. Sulla curva di Amoroso per la distribuzione dei redditi.** Scritti matematici in onore di Filippo Sibirani, pp. 181–193. Cesare Zuffi, Bologna, 1957.

Graduation of two numerical examples of income distributions by Amoroso's formula

$$f(x)=c(x-h)^{(\gamma-\varepsilon)/\varepsilon}\exp\{-\gamma(x-h)^{1/\varepsilon}\}$$

[Ann. Mat. Pura Appl. (4) **2** (1925), 123–159].

P. Johansen (Copenhagen).

Watanabe, Yoshikatsu; Wajiki, Isamu; and Kawashiro, Teruaki. On the compound normal distributions. J. Gakugei Tokushima Univ. Nat. Sci. Math. **7** (1956), 53–65.

The author has examined the compound normal distribution and especially given conditions that the resulting compound distribution is again a normal distribution.

A. Jensen (Copenhagen).

Haldane, J. B. S. A class of efficient estimates of a parameter. Bull. Inst. Internat. Statist. **23** (1951), part II, 231–248.

Consider trials capable of producing one of the m possible results R_k with a fixed probability $f_k(\xi)$, depending upon a parameter ξ. The functions $f_k(\xi)$ are supposed continuously differentiable. The symbol n_k is used to denote the number of occurrences of R_k in the course of N independent replications of trial. The problem considered is that of determining a class of estimates of ξ that are all asymptotically consistent and efficient. For an integer $k\neq 0$ the estimate x_k of ξ is obtained by expression D_k defined as follows. If $k\neq -1$, then

$$D_k=\frac{(N+k)!}{N!}\sum_{m}^{r=1}f_r^k(x)\,\frac{(n_r+k)!}{n_r!}.$$

On the other hand, $D_{-1}=-N^{-1}\sum n_r\log f_r(x)$. Thus x_{-1} is the maximum likelihood estimate of ξ. — Of all the expressions D_k, the author prefers D_1, described as the discrepancy. In addition to showing that all the estimates x_k are asymptotically consistent and that they have the same asymptotic variance, the author evaluates the bias of the estimates and gives asymptotic formulae for the first four cumulants. *J. Neyman* (Berkeley, Calif.).

Broadbent, S. R. Examination of a quantum hypothesis based on a single set of data. Biometrika **43** (1956), 32–44.

Given a set of n measurements $y_1, y_2, \cdots, y_n$ of n physical magnitudes of the same kind, for example of energy levels of n nuclei, it is desired to test the hypothesis H, suggested by the data on hand, that the kth

measurement may be represented by the equation $y_k=A+2Br_k+\varepsilon_k$, where ε_k stands for a random error of measurement, A and B unknown constants and r_k an integer number. Here $2B$ stands for the "quantum" and the hypothesis H is described as the quantum hypothesis. In the present paper it is assumed that $A=0$ and the problem of testing H is considered with respect to the alternative that the y's are uniformly distributed between certain limits. The contemplated method of solution is based on the relation between quantities $(\frac{1}{3}-s^2/b^2)\sqrt{n}$ and $(\max y)/b$, where b is the value of B that minimizes $\sum(y_i-2r_iB)^2$, with appropriately selected integers r_i, and s^2 the corresponding minimum of the sum. The details of the test procedure are worked out using the results of an extensive random sampling experiment. *J. Neyman* (Berkeley, Calif.).

Diederich, Gertrude W.; Messick, Samuel J.; and Tucker, Ledyard R. A general least squares solution for successive intervals. Psychometrika **22** (1957), 159–173.

Given n stimuli sorted into $k+1$ categories (by use of k category boundaries) on some attribute continuum, with experimentally determined cumulated proportions (p_{ig}) for the successive categories for each stimulus and corresponding normal deviate values (z_{ig}). The experimental model is

$$t_g=m_i+s_iz_{ig}\ (g=1, 2, \cdots, k;\ i=1, 2, \cdots, n).$$

The investigator wishes to estimate the upper boundaries of the categories (t_g), the mean scale values for the stimuli (m_i) and their dispersions (s_i).

The authors present an iterative procedure to estimate the t_g, m_i and s_i based on minimizing $\sum_i\sum_g w_{ig}(m_i+s_iz_{ig}-t_g)^2$, subject to restrictions that the weighted mean and variance of the t's are certain values, where the weights (w_{ig}) are given. Apparently the authors were unaware that usual least squares regression procedures, including the Abbreviated Doolittle and square root methods, could be used by solving for all constants in terms of one of the t's, say t_k, i.e. let t_k take the place of the dependent variable. Then t_k can be determined by use of the restriction on the weighted variance of the t's.

An example is given for $n=4$, $k=3$. *R. L. Anderson.*

Lev, Joseph; and Kinder, Elaine F. New analysis of variance formulas for treating data from mutually paired subjects. Psychometrika **22** (1957), 1–15.

The experimental design considered in this paper is one in which each of a group of several subjects is observed in the presence of each of the other subjects of the group. The entire sets of possible pairings are repeated or replicated on several occasions. The data is concerned with the seven chimpanzee subjects each of which is paired with every other subject in the protagonist-partner relationship, making 42 pairings in each series of observations. Analysis of variance formulas are described under the parametric model and mixed one.

T. Kitagawa (Princeton, N.J.).

Blackwell, David; and Hodges, J. L., Jr. Design for the control of selection bias. Ann. Math. Statist. **28** (1957), 449–460.

Suppose an experimenter E wishes to compare the effectiveness of two treatments A and B on a somewhat vaguely defined population. As individuals arrive E decides whether they are in the population and if they are, he administers A or B according to a scheme prepared by a statistician S.

Assume now that the experimenter wishes to bias the experiment to make it appear that A is favorable, by selecting "favorable" persons to receive A and "unfavorable" persons to receive B. His problem is to guess before selection what is the next treatment to be administered. Conversely the statistician wishes to design the scheme of administering A or B to eliminate this bias.

It is shown that this design problem can be formulated as a two-person game. On each move each of the players E and S privately selects one of the letters A or B with the restriction that exactly n of S's choices must be A. They then compare selections; if they agree S pays E one unit. The main result is that S's optimum strategy is to select A or B randomly with equal probability until either the number of A's or the number of B's equal n. Even with this optimum strategy of S there is still a large selection bias and alternative designs are considered. *D. G. Chapman* (Seattle, Wash.).

Page, E. S. Continuous inspection schemes. Biometrika **41** (1954), 100–115.

Überall, wo Beobachtungen in geordneter Reihenfolge gewonnen werden, kann es vorkommen, daß die Gesamtheit der Beobachtungen in Untergesamtheiten aufzuteilen ist, deren jede als Zufallsstichprobe aus einer gemeinsamen Verteilung, jede Untergesamtheit jedoch einem anderen Wert des Parameters (z. B. Mittelwert) dieser Verteilung entsprechend, angesehen werden kann. Die in der Arbeit behandelten Problemstellungen betreffen die evtl. Identifizierung der Untergesamtheiten bzw. die Aufdeckung der Wechsel des Parameterwertes. In der Praxis ergeben sich entsprechende Probleme etwa bei psychologischen Testversuchen, bei denen sich der Anteil richtiger Entscheidungen im Laufe des Experiments systematisch ändert, sowie im Aufgabenbereich der statistischen Qualitätskontrolle in der Industrie, wobei es darum geht, Wechsel in der Qualität der Ausbringung kontinuierlicher Produktionsprozesse zu erkennen. Die Terminologie dieses Anwendungsgebietes wird wegen ihrer besseren Verständlichkeit deshalb auch in der Arbeit bevorzugt. *G. Wünsche* (Zbl **56** (1955), 380).

★ **Bowker, Albert H. Continuous sampling plans.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol.V, pp. 75–85. University of California Press, Berkeley and Los Angeles, 1956. $5.75.

Expository paper on rectifying sampling of a continuous production, where the inspection dichotomizes articles as defective or nondefective. A relevant study not alluded to is that of the paper reviewed above.

F. J. Anscombe.

Kamat, A. R. A sampling experiment to verify the approximate theoretical distributions of mean successive difference. Calcutta Statist. Assoc. Bull. **7** (1956), 17–26.

The sampling experiment is carried out with the help of the 25,000 random normal deviates given in Wold's table [Random normal deviates, Cambridge, 1948; MR **10**, 553] in order to investigate the distribution of the statistic

$$d=\sum_{i=1}^{n-1}|x_i-x_{i+1}|/(n-1).$$

The 2500 independent values of D are obtained for $n \leq 10$. The empirical distributions of D were extracted from the experiment for the various values of n: (a) 3, 4, 6, 8, 10; (b) 12, 15, 18, 20; (c) 25, 30; (d) 40, 50; (e) 100. Then the author compares each of the empirical distributions respectively with that of his proposed Pearsonian approximations given in Biometrika (1953), 116–127 [MR 15, 451]. *T. Kitawaga* (Princeton, N.J.).

Cansado, Enrique. Sampling without replacement from finite populations. Trabajos Estadist. 8 (1957), 3–12. (Spanish summary)

Suppose a finite population consists of N distinct and identifiable elements. A sample of size n ($2 \leq n \leq N$) is obtained in n successive draws of one sampling unit each, without replacing the units selected in the previous draws and a known probability of selection being assigned to each of the remaining elements of the population. Let X_i be some numerical measurement associated with the ith sampling unit and let $T=\sum_{i=1}^{N} X_i$. Unbiased estimates of T and of the variance of the estimate of T were obtained earlier by Narain [J. Indian Soc. Agric. Statist. 3 (1951), 169–174; MR 13, 570], Horvitz and Thompson [J. Amer. Statist. Assoc. 47 (1952), 663–685; MR 14, 777] and Yates and Grundy [J. Roy. Statist. Soc. Ser. B. 15 (1953), 253–261] under the assumption that the probability of selecting from among the available units at any draw is proportional to the probability of selecting it at the first draw. The author has shown, by following straightforward methods, that the results hold in general and the only assumption needed is that every pair of sampling units has a positive probability of being included in the sample. There are some typographical errors.

O. P. Aggarwal (Edmonton, Alta.).

Ogawara, Masami. An exact test for moving averages. Bull. Math. Statist. 7 (1957), 77–83.

An adaptation of the F test. *H. Wold* (Uppsala).

Lomnicki, Z. A.; and Zaremba, S. K. On the estimation of autocorrelation in time series. Ann. Math. Statist. 28 (1957), 140–158.

The authors investigate the estimation of the autocovariances of a linear process of general type with a polynomial trend. Asymptotic expressions are obtained for the bias, variance and covariances between two such estimates. Using a method of "statistical differentials" the authors arrive at expressions for the asymptotic first and second order moments of estimates of the autocorrelation coefficients of the process. *U. Grenander.*

See also: Numerical Methods: Sibuya and Toda. **Computing Machines:** Lipton. **Probability:** Barra; Parzen. **Astronomy:** Ramakrishnan; Nahon. **Economics, Management Science:** Vegas. **Biology and Sociology:** Haskey; Kempthorne. **Information and Communication Theory:** Van Meter and Middleton.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Tzénoff, Iv. Sur une transformation des équations du mouvement des systèmes matériels. C. R. Acad. Bulgare Sci. 9 (1956), no. 3, 5–8. (Russian summary)

The author had previously established the equations of motion of a holonomic system with n degrees of freedom in the form

$$\tfrac{1}{2}\frac{\partial \ddot{T}}{\partial \dot{q}_i}=Q_i+\tfrac{1}{2}\frac{\partial T}{\partial q_i} \quad (i=1, \cdots, n),$$

where the notation is the standard one used in the Lagrangian theory. By using a modified virtual work $\delta w=\sum_i (Q_i+\frac{1}{2}\partial T/\partial q_i)\delta q_i$, the equations can be written in a very simple form. By subjecting the $\dot{q}_1, \cdots, \dot{q}_n$ to non-integrable restraints, it is also possible to treat nonholonomic systems in a simple way. The present paper is concerned with a transformation theory of these simplified forms, when the $\dot{q}$'s are expressed linearly in terms of new variables $\omega_1, \omega_2, \cdots$ and then eliminated.

D. C. Lewis (Baltimore, Md.).

Storchi, Edoardo. Una soluzione delle equazioni indefinite della meccanica dei continui negli spazi riemanniani. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. 90 (1956), 369–378.

It is known that (*) $\Phi^{ij}=R^{ij}-\frac{1}{2}Ra^{ij}$, where R^{ij} is the contracted Riemann tensor and R is the total curvature, is a solution of the system (**) $\Phi^{ij}{}_{ij}=0$ in any Riemannian space of dimension $n>2$. Guided by this fact and by Finzi's general solutions of (**) for spaces of constant curvature when $n=2$, 3, or 4, the author finds by trial certain solutions of (**) for arbitrary Riemannian spaces when $n=2, 3, 4$. These solutions, while not general for the spaces considered, involve an arbitrary function of the total curvature; for $n=3, 4$, they include (*) as a special case. The method, which consists in adding modifying terms to Finzi's solution, is considered by the author to be applicable for arbitrary n. *C. Truesdell.*

Rasković, D. Procédé vectoriel de détermination de coordonnées sphériques des vecteurs de vitesse et d'accélération. Srpska Akad. Nauka. Zb. Rad. 50, Mat. Inst. 5 (1956), 249–263. (Serbo-Croatian. French summary)

En mécanique rationnelle on détermine habituellement les projections des vecteurs de vitesse et d'accélération sur les axes d'un système de coordonnées sphériques par des coordonnées généralisées à l'aide de l'équation de Lagrange.

Dans ce travail on donne pour cette détermination, une méthode vectorielle simple, en n'utilisant que la formule de rotation dans le plan, respectivement le principe du mouvement relatif dans l'espace. La méthode proposée est très pratique, car elle permet de saisir le sens physique des différentes parties de ces projections. On donne, en outre, l'interprétation des coordonnées covariantes et contravariantes de ces vecteurs.

Résumé de l'auteur.

Sergeev, V. I. Some questions arising in calculating the accuracy of gyroscopic verticals. Trudy Inst. Mašinoved. Sem. Točn. Mašinostro. Priborostr. 8 (1955), 64–79. (Russian)

Accuracy of gyroverticals is determined by probabilistic analysis of small oscillations of gyromotor rotor relative to verticals (astatic gyros with pendulum correction) mounted on an aircraft moving uniformly in a straight line. Moments due to acceleration of aircraft motion acting on gyrovertical; to gravity; to friction;

to proportional correction moments acting on the gyrovertical are analyzed. Formulas are given to compute the mathematical expectation and variance of the precession angle for gyroverticals free and with linear correction. Recommendations are given to decrease the variance and the kind of regulation permitting reduction of the mathematical expectation to zero is indicated.
M. D. Friedman (Cambridge, Mass.).

Schmieden, C. Nichtlineare Schwingungen bei zwei Freiheitsgraden. I. Ing.-Arch. **25** (1957), 292–302.
Contrary to Pöschl's procedure the author in this paper treats the simplified mathematical problem of the two nonlinear differential equations

$$\ddot{\theta}+\alpha^2\ddot{\varphi}+\omega_0^2\theta+\beta\omega_0^2\theta^2=0,\quad \ddot{\theta}+\ddot{\varphi}+\omega_0^2\varphi+\beta\omega_0^2\theta^2=0$$

which govern the oscillations of a composite pendulum with $l_1=l_2$, $\alpha^2=m_2/(m_1+m_2)$, where $0<\alpha^2<1$. The periodic solutions of the above system of differential equations are assumed to be power series. The following cases are treated in detail: $\alpha=4/5$, $\alpha=3/5$ and the periodic solutions in the limiting case when α is very small.
D. P. Rašković (Belgrade).

Bradistilov, G. Sur les solutions périodiques et asymptotiques du mouvement autour de l'état d'équilibre d'un système de *N*-pendules physiques successivement liés dans un plan. C. R. Acad. Bulgare Sci. **8** (1955),no. 4, 5–8. (Russian summary)

Chebotarev, G. A symmetrical orbit for a flight around the moon. Byull. Inst. Teoret. Astr. **6** (1957), 487–492. (Russian. English summary)
In this article the orbit of a rocket for the flight around the moon is calculated. By means of numerical integration the effects of the earth's and moon's gravitational fields are taken into account. No fuel expense is required during the trip. *Author's summary.*

See also: Differential Geometry: Lagally. **Elasticity, Plasticity:** Fung. **Relativity:** Arzeliès.

Statistical Thermodynamics and Mechanics

★**Frenkel, J. I. Statistische Physik. Aus dem Russischen übersetzt von H. Jancke.** Akademie-Verlag, Berlin, 1957. xxii+783 pp. DM 48.00.
In 1948 Frenkel published a second, enlarged edition [Izdat. Akad. Nauk SSSR, Moscow] of his 1932 treatise on statistical physics, and the book here reviewed is the German translation of this second edition. The term statistical physics, very common in the Russian physical literature, is somewhat more general than what Western physicists would call statistical mechanics or statistical thermodynamics. It denotes quite generally that part of physics which deals with systems composed of many identical particles, whatever their sort. Actually, in view of the growth of the subject far beyond the original fields of statistical mechanics and thermodynamics the very general name of statistical physics has much to commend it.
In many respects Frenkel's book is influenced by the personal interests of the author, whereby the choice and ordering of topics is rather unconventional for the present day reader. Whereas a number of recent developments remain essentially untouched, a deepgoing and often very interesting discussion is given of points which are usually treated more briefly. The author devotes much space to a presentation of his views on the foundations of statistical mechanics. Although no essentially new conclusions are reached, this presentation of the problem by an outstanding physicist of the older generation is very much worth reading. It has furthermore the advantage of great clarity.
The book is divided in three parts of about equal length. The first part deals with the general principles and methods of classical statistical physics. Half of it concerns ideal or very dilute gases and follows the kinetic approach according to Boltzmann. This case serves as a model to introduce ensemble theory. A very careful discussion is given for the meaning of the principles of the Boltzmann and Gibbs methods; the author also stresses the degree of rigor (or lack of it) in some of the key derivations. Much is said about fluctuations and the identity between exceptionally large spontaneous fluctuations and macroscopic non-equilibrium situations, but no word is devoted to the fundamental work of Onsager on irreversible thermodynamics and the decay of fluctuations, in which this identity plays a central role.
The second part, entitled special methods and problems of classical statistical physics, opens with a chapter on systems with long range forces. The theory of real gases is then given. The author bases his discussion on the van der Waals equation and completely leaves out the Ursell-Mayer expansion. Solutions, diffusion and dissociation processes are treated. Solids for not too low temperatures are analyzed in one dimension and, for three dimensions, in the Debye continuum approximation. A chapter is devoted to melting.
The last part of the book, devoted to quantum statistics, opens with an introduction to quantum mechanics. The general principles of quantum statistics are given and applied to various topics: quantum gases, solids for low temperatures (Debye theory), photon gas, magnetism, statistical theory of atomic nuclei, liquid helium at low temperature. Here again many important developments are not taken into account, which is perhaps regrettable in a book of this size. *L. Van Hove* (Utrecht).

Buff, Frank P.; and Saltsburg, Howard. Curved fluid interfaces. III. The dependence of the free energy on parameters of external force. J. Chem. Phys. **26** (1957), 1526–1532.
[For parts I and II see same J. **25** (1956), 146–153; **26** (1957); 23–31, MR **18**, 94.]
The properties of interfaces in a multi-phase fluid subject to an external field are investigated using statistical thermodynamics. The formulas include correction terms to the free energy which define the range of validity of the classical expressions used in thermodynamics. It is also shown how the location of the interfaces is determined by the shape of the container. *G. Newell.*

Müller, Henning. Zur Frage der stationären Bewegung eines einfachen Makromolekülmodells im strömenden Medium. Ann. Physik (6) **19** (1956), 304–308 (1957).

Price, P. J. The linear Hall effect. IBM J. Res. Develop. **1** (1957), 239–248.
A new method for handling the Boltzmann equation is used to obtain an exact, general, formula for the linear Hall effect in an electronic conductor. Expressions for the ordinary and the quadratic magneto-conductivity

are obtained also. These formulas involve a vector mean free path related to the electron velocity by an integral equation; the solution of this equation is discussed, and perturbation and variational methods of approximation are proposed. The conditions for the validity of previous theories are shown. (Author's abstract slightly changed.)
P. W. Anderson (Murray Hill, N.J.).

Buckingham, A. D. The influence of a strong magnetic field on the dielectric constant of a diamagnetic fluid. Proc. Phys. Soc. Sect. B. **70** (1957), 753–760.

The effect of a strong magnetic field on the static dielectric constant of a diamagnetic fluid is investigated using statistical mechanical methods; a formula is derived relating the dielectric constant ε to the square of the field strength. The theory indicates that the change in ε is very small in non-polar substances (although it is detectable in magnetic double refraction experiments), but should be measurable in polar liquids; it is known to be substantial in 'liquid crystals', and it is suggested that it might also be large in solutions of large molecules. In a polar substance the chief contributor to the change in ε is related to the anisotropy in the diamagnetic susceptibility tensor and is proportional to the square of the molecular electric dipole moment. Relationships between the effect and electric and magnetic double refraction are established. Experimental data is scanty, but it is indicated how observations of the effect could lead to information about molecular properties.
Author's summary.

Ekstein, H. Ergodic theorem for interacting systems. Phys. Rev. (2) **107** (1957), 333–336.

The ergodic theorem of von Neumann for quantum-mechanical systems is recast to apply to a single system as small as one likes in weak interaction ($\lambda \to 0$) with a large ($N \to \infty$) number of similar systems. This may correspond more closely to the usual applications in statistical mechanics. *P. W. Anderson.*

Poirier, Jacques C. Integral equation method for determining approximate fluid distribution functions. J. Chem. Phys. **26** (1957), 1427–1439.

The n-particle position distribution functions $R_N^{(n)}$ are expressed as multiple Taylor series in n coupling parameters $\lambda_1 \cdots \lambda_n$, where the magnitude of λ_i measures the extent to which the ith particle interacts with the other particles in the system. An infinite sequence of integro-differential equations for the coefficients of the series is then obtained directly from the basic Boltzmann equation for $R_N^{(N)}$. To render the sequence finite, the series may be cut off at terms of some suitable order in the λ_i. Ending the series with terms of second order in the λ_i leads to the superposition approximation. In the two special cases of point charges and particles interacting with an exponential law of force, it is shown that extending the series to include third order terms does not introduce any corrections to the second order results.
S. Prager (Minneapolis, Minn.).

Prigogine, I.; et Balescu, R. Sur la théorie moléculaire du mouvement brownien. Physica **23** (1957), 555–568.

Applying the theory of relaxation in weakly coupled systems developed by Brout and Prigogine [Physica **22** (1956), 35–47], the authors discuss in detail the approach to equilibrium of (1) the energy distribution among a group of normal modes interacting through small anharmonic terms in the potential, and (2) the velocity distribution among a group of particles interacting through small central forces. Only the behaviour of a single mode or particle is considered, the others being assumed in equilibrium with a constant temperature reservoir. Fokker-Planck equations are developed which permit the calculation of the energy and velocity distributions as a function of time, and mean values as well as mean square deviations are obtained. It is shown that the relaxation is a non-gaussian Markoff process, although in case (2) it becomes gaussian if the velocity distribution is assumed to be out of equilibrium only with respect to the orientation of the velocity vector. *S. Prager.*

Prigogine, I.; and Philippot, J. On irreversible processes in non-uniform systems. Physica **23** (1957), 569–584.

The work of Brout and Prigogine [Physica **22** (1956), 621–636] is extended to systems in which the initial distribution depends upon the angle as well as the action variables. The results are applied to the weakly coupled oscillators treated in the paper reviewed above to obtain a Boltzmann equation for the time dependence of the distribution function. In addition to the usual flow and collision terms, the equation also involves terms corresponding to interaction between flow and collision. Wave propagation, relaxation, and thermal conduction in anharmonic crystals are also treated. *S. Prager.*

Prigogine, I.; and Henin, F. On the general perturbational treatment of irreversible processes. Physica **23** (1957), 585–596.

This paper extends the results obtained by Prigogine and Brout [Physica **22** (1956), 621–636] for weakly coupled systems to include terms of third and fourth order in the coupling parameter, permitting application to systems where the coupling is sufficiently strong to affect the equilibrium properties. The equations for the time rates of change of the Fourier components of the density function no longer form a diagonal set, and cannot be considered independently of each other. An appendix treats the effect of time dependent external forces. *S. Prager* (Minneapolis, Minn.).

Haase, Rolf. Der Zweite Hauptsatz der Thermodynamik und die Strukturbildung in der Natur. Naturwissenschaften **44** (1957), 409–415.

Popoff, Kyrille. Thermodynamique des processus irréversibles. C. R. Acad. Sci. Paris **245** (1957), 925–926.

Elasticity, Plasticity

Barta, J. On the non-linear elasticity law. Acta Tech. Acad. Sci. Hungar. **18** (1957), 55–65. (German, French and Russian summaries)

On the basis of certain assumptions and approximations, the author deduces restrictions on the stress-strain relations for isotropic materials. The paper is not free of logical errors, one being encountered in his invalid proof of existence of a strain energy function. *J. L. Ericksen.*

Ugodchikov, A. G. On the solution of the plane problem for a composite isotropic medium by means of electrical modelling of the conformal transformation. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 343–347. (Ukrainian. Russian and English summaries)

The problem of the numerical solution of the plane

problem of the theory of elasticity by means of functions of a complex variable is presented in this paper. The function $z=\omega(\zeta)$, indispensable for this, which transforms the unit circle $|\zeta|\leqq 1$ into the region S of the z-plane, is calculated approximately by means of electrical modelling of the conformal mapping. *Author's summary.*

Gubenko, V. S. Some contact problems of the theory of elasticity and fractional differentiation. Prikl. Mat. Meh. **21** (1957), 279–280. (Russian)

★ **Alblas, Johannes Bartholomeus. Theorie van de driedimensionale spanningstoestand in een doorboorde plaat. [Theory of three-dimensional stress systems in plates with holes.]** H. J. Paris, Amsterdam, 1957. xii+127 pp.

This dissertation deals with two problems in the classical theory of elasticity, both of which have received previous attention: the three-dimensional analysis of the stresses, (A) due to stretching and (B) due to flexure, in a plate of infinite extent and arbitrary thickness, which has a transverse cylindrical hole. Earlier three-dimensional investigations of the plane problem (A) are due to Green [Philos. Trans. Roy. Soc. London. Ser. A. **240** (1948), 561–597; MR **9**, 547], as well as to Sternberg and Sadowsky [J. Appl. Mech. **16** (1949), 27–38; MR **10**, 652] who employed a variational technique. Later on, Green [Proc. Roy. Soc. London. Ser. A. **195** (1949), 533–552; MR **11**, 286] simplified and extended his approach, which rests on the rigorous satisfaction of all field equations, thus arriving at a systematic scheme for the treatment of general plate problems of type (A) and (B). It is this scheme which forms the basis of the present investigation.

The method adopted by the author involves the construction — with the aid of harmonic displacement potentials — of an infinite series of component solutions, each term of which leaves the plate-faces free from tractions. The coefficients of superposition are then determined from the boundary conditions for the lateral boundary; this leads to a doubly infinite system of linear algebraic equations, which is solved by the usual truncation process. The asymptotic behavior of the solutions reached, in the neighborhood of zero plate-thickness, is studied in detail. The monograph under discussion represents a particularly valuable contribution by virtue of its wealth of numerical results, which permit a more complete appraisal of the status of the corresponding two-dimensional solutions than had heretofore been available. In connection with (A), comparisons are made with the associated generalized plane stress and plane strain solutions; the solution to (B) is discussed in its relation to classical plate theory and to the refined approximate plate theory of E. Reissner [J. Appl. Mech. **12** (1945), A-69–A-77; MR **7**, 42].

As pointed out by the author, the convergence of the solutions obtained deteriorates as the ratio of the plate-thickness to the diameter of the hole tends to infinity. An adequate investigation of the interesting "boundary-layer effects" near the rim of the hole, in a relatively thick plate, would require separate attention and, to date, appears to have escaped successful analysis.

E. Sternberg (Providence, R.I.).

Manacorda, Tristano. Relazioni fra deformazione e stato di tensione per un generico solido incomprimibile a trasformazioni reversibili. Boll. Un. Mat. Ital. (3) **12** (1957), 1–8.

The author considers as his starting point the usual stress-strain relations for finite deformation of an incompressible elastic solid. Writing $\mathscr{F}_r$ for the elastic potential regarded as a function of the principal strains Γ_h and of three parameters φ_h expressing the rotation of the principal axes of strain, he transforms the stress-strain relations into a form where the effects of strain and rotation are separated. In addition to the expected relation between principal stresses and principal strains, he finds that

$$(*) \qquad \rho\frac{\partial\mathscr{F}_r}{\partial\varphi_h}=-\sum_{l=1}^{3}2t^{l+3}\omega_{hl}(\Gamma_{l+1}-\Gamma_{l+2}),$$

where the t^{l+3} are the shear stresses and where the ω_{hl} are certain rotation components. A body is defined as isotropic if $\mathscr{F}_r$ is a function of the Γ_h only. From (*) it follows that the body is isotropic if and only if all t^{l+3} vanish, i.e., if and only if the principal axes of stress coincide with the principal axes of strain. *C. Truesdell.*

Jindra, F. Beitrag zur nichtlinearen Torsion. Österreich. Ing.-Arch. **11** (1957), 134–146.

This is a continuation of the author's work [Z. Angew. Math. Phys. **6** (1955), 345–355; Ing.-Arch. **23** (1955), 122–129, 373–378; MR **17**, 801, 315] on a special theory of non-linear elasticity valid only for infinitesimal strain. He obtains solutions for the torsion of rods with circular, elliptic, triangular, and quadratic cross-sections.

W. Noll (Pittsburgh, Pa.).

Csonka, P. Méthode de calcul numérique des contraintes causées par la torsion. Acta Tech. Acad. Sci. Hungar. **18** (1957), 399–407. (German, English and Russian summaries)

Holman, D. F. A finite series solution for grillages under normal loading. Aero. Quart. **8** (1957), 49–57.

A method has been given by Klitchieff [Bull. Acad. Serbe Sci. Cl. Sci. Tech. (N.S.) **2** (1951), 69–76, 77–82; MR **13**, 404] for analysis of grillages of intersecting beams under lateral loading. Klitchieff's solution was in the form of infinite Fourier series and the expressions for bending moments converged approximately as $\sum n^{-2}$, making numerical computation difficult. This paper shows that, for grillages in which one set of beams are evenly spaced, Klitchieff's solution can be adapted to give finite series for deflections and bending moments at the intersections of the beams. The number of terms in the series is not greater than the number in the equally spaced set (referred to in this paper as set *A*). A numerical example is given and an accurate solution is obtained with appreciably less computation than other known methods would entail. *Author's summary.*

Koiter, W. T.; and Alblas, J. B. On the bending of cantilever, rectangular plates. IV. Nederl. Akad. Wetensch. Proc. Ser. B. **60** (1957), 173–181.

The authors present and discuss numerical results of an analysis described by them in parts I–III [same Proc. **57** (1954), 250–258, 259–269, 549–557; MR **17**, 685] relevant to the bending of a semi-infinite elastic cantilever strip under a uniform edge moment at infinity. The calculations are contrasted with those which correspond to a more conventional treatment of the problem.

F. B. Hildebrand (Cambridge, Mass.).

Layrangues, P. Calcul des contraintes de membrane dans un voile mince de révolution. Applications aux réservoirs et tours dont la surface médiane est un hyperboloïde de révolution à une nappe soumis à une pression normale. Coupoles elliptiques. Effet du vent sur ces constructions et sur les tours tronconiques. Ann. Ponts Chaussées **127** (1957), 517–547.

This article deals with the determination of the factors of static equilibrium of a thin shell in the form of a surface of revolution subjected to continuous loads.

From the general vector equation of equilibrium in conjugate rectangular coordinates projected on the axis of revolution and the three intrinsic equations of equilibrium, the author derives a very simple basic system which readily lends itself to the study of concrete cases.

Thus, from these relations, he derives the equations of equilibrium of a tower in the form of a single sheet hyperboloid of revolution subjected to any given continuous normal load. He shows that in this case, by suitable selection of the functions entering into the stress components and of the independent variable characterizing the meridian line, the problem consists of integrating a conventional system of 2 equations with partial linear derivatives with constant coefficients, covering two of the stress functions. *From the author's summary.*

Aĭnola, L. Ya. Variational problems in the non-linear theory of elastic shells. Prikl. Mat. Meh. **21** (1957), 399–405. (Russian)

In order to obtain variational problems in the non-linear theory of shells use is made of the method of transformation of variational problems in a generalized form (see, e.g., Courant-Hilbert, Methods of mathematical physics, vol. I, Chap. IV, Interscience Publishers, Inc., New York, 1953; MR **16**, 426). By this method there is introduced a cycle of fundamental variational problems of the simplified non-linear theory of shells with simple boundary conditions. At the end of the article the possibility is indicated of obtaining a series of other variational problems. *Author's summary.*

Karas, K. Die Auswölbungen der Kreis- und Kreisringmembranen unter hydrostatischem Druck. I. Homogener Spannungszustand. Ing.-Arch. **25** (1957), 359–380.

Hedgepeth, John M. Flutter of rectangular simply supported panels at high supersonic speeds. J. Aero. Sci. **24** (1957), 563–573, 586.

Title problem is dealt with on the following basis: The plate behavior is described by small-deflection, thin-plate theory; midplane stresses (chordwise and spanwise compression are of principal interest) are taken into account. The pseudo-static approximation is used to define the aerodynamic mechanism, which author claims should be adequate for Mach numbers in the range from somewhat under 2 to higher values.

For the simply-supported, rectangular plate, a modal solution is employed, using a truncated double-trigonometric series to describe the plate deflection. The number of terms to be used in the series is estimated from some preliminary studies of the two-dimensional problem (infinite span); in the latter case, the differential equations can be solved exactly and compared with the results of approximate methods.

While the author is careful and ingenious in his work, the difficulty of the problem prevents any convincing proof of convergence of the modal approach. *M. Goland.*

Saharov, I. E. Frequencies of eigen-vibrations of ring-shaped plates. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 5, 107–113. (Russian)

For the first three eigenvibrations of a ring-shaped plate, the author gives in graphical form the results of some calculations of the variation of frequency with the magnitude of the relative opening under the typical boundary conditions of clamped edge and hinged edge. *E. F. Beckenbach* (Los Angeles, Calif.).

Fung, Y. C. Some general properties of the dynamic amplification spectra. J. Aero. Sci. **24** (1957), 547–549.

This brief note discusses the dynamic amplification spectrum of a linear oscillator of one degree of freedom subject to an impulsive force. The behavior of the amplification spectrum for very small and very large frequencies is studied. The inverse problem of determining the loading pulse from an observed amplification spectrum is also raised, although not solved. *W. E. Boyce.*

Belyaeva, G. M. Application of the theory of perturbation to the problem of vibrations of a beam. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 1, 11–21. (Russian)

The problem of vibrations of a light beam with concentrated masses is very important in engineering. In general, the problem of a homogeneous beam with equal masses reduces to a finite-difference equation of fourth order with constant coefficients. But, if a beam is non-homogeneous and carries unequal masses, the problem becomes more difficult. In this case, following the Courant-Hilbert scheme, the author uses the theory of perturbations to obtain the eigenvalues and eigenvectors of the finite-difference equation with variable coefficients. The system of the amplitude equations is given in matrix form with a frequency equation. The problem of a free beam is discussed, as well as the problem of the simply supported beam. Two examples are given. Finally, a proof of the convergence of the process of successive approximations is given. *D. P. Rašković* (Belgrade).

Martin, A. I. Approximation for the effect of twist on the vibration of a turbine blade. Aero. Quart. **8** (1957), 291–308.

Modern turbine blades are generally twisted about their axes, the shape resembling that of a helicoid. This makes the determination of their overtone frequencies much more difficult. The equations governing the flexural vibrations may be obtained from the Clebsch-Kirchhoff theory for rods which are naturally twisted. By using these equations, this paper considers a first-order approximation for the effect of uniform twist. *Author's summary.*

Ray, K. D. Dynamics of the vibration of a bar excited by transverse impact of an elastic load: displacement at any point during impact. Indian J. Theoret. Phys. **4** (1956), 65–70.

Masur, E. F.; and Milbradt, K. P. Collapse strength of redundant beams after lateral buckling. J. Appl. Mech. **24** (1957), 283–288.

Upper and lower bounds are found for the maximum load carrying capacity of a beam which fails by purely elastic buckling. Finite slope but small strain theory is employed [see Th. v. Kármán, Encykl. Math. Wiss., Bd. IV, T. 4, Teubner, Leipzig, 1910, p. 311] and the

supports are supposed to introduce no additional axial force during buckling. The theorems established are analogous to those of plastic limit analysis and design. They are: If a set of moments and reactions can be found which equilibrates a given system of loads, and if the equilibrium so defined is stable, then the structure is safe under the loads. If a collapse mode can be found which, for a given system of loads, corresponds to a decrease in the potential energy, the structure is unsafe. An upper bound does not exist in all cases, the load may increase indefinitely as buckling progresses. Some experimental results are compared with the theory.

D. C. Drucker (Providence, R.I.).

Bodner, S. R. General instability of a ring-stiffened, circular cylindrical shell under hydrostatic pressure. J. Appl. Mech. **24** (1957), 269–277.

Assuming that the ring stiffeners are equally spaced and the wave length of the deformed shell includes a numbers of stiffeners, the behavior of a ring-stiffened shell is approximated by an equivalent orthotropic shell. The equilibrium equations are derived by variational methods from the strain energy and are presumably as accurate as those of Donnell [NACA Tech. Rep. no. **479** (1934)] for the isotropic case, i.e., they are better for shorter, thinner shells. The shell is assumed to be simply supported at its extreme ends.

The axial extensional and bending rigidities and the shear rigidity are taken to be the same for the orthotropic shell as for the isotropic since the orthotropy itself is a model for the effects of ring-stiffening. Two constants in the basic differential equation must still be determined. One is determined from the effective thickness of the shell. The second depends on the effective length in bending of shell plating for which there is no rigorous analysis. An approximate form is given by requiring appropriate limiting behavior. The author points out that this difficulty appears intrinsically in the selection of functions for a Rayleigh-Ritz procedure. Numerical calculations show a close correlation with experiment.

G. H. Handelman (Troy, N.Y.).

Pflüger, Alf. Das Beulproblem der orthotropen Platte mit Hohlsteifen. Z. Flugwiss. **5** (1957), 178–181.

The overall buckling of a plate with stiffeners (of closed hollow section) in the direction of compression is considered. By appropriate physical assumptions the resultant forces and moments in the stiffeners and plate are expressed in terms of the displacements of the middle surface of the plate and equations of equilibrium set up.

Numerical results show that the torsional rigidity of the stiffeners has considerable influence on the buckling stress.

R. C. T. Smith (Armidale).

Klyušnikov, V. D. Stability of plates at the elastic limit. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 7, 41–56. (Russian)

Eringen, A. Cemal. Elasto-dynamic problem concerning the spherical cavity. Quart. J. Mech. Appl. Math. **10** (1957), 257–270.

This paper treats the behavior of an infinite isotropic elastic material acted upon by arbitrary dynamical tractions on the surface of a spherical cavity. A Fourier transform technique is used, and the solution of the transformed problem is expressible as an infinite series whose terms involve the half-order Hankel functions of the first kind. Several particular problems are discussed with primary attention given to blast type loading. In this case the inverse transform is carried out by numerical integration; curves are presented showing the radial displacement, radial stress, and hoop stress as functions of the radial distance.

W. E. Boyce (Troy, N.Y.).

Mann-Nachbar, P. The interaction of an acoustic wave and an elastic spherical shell. Quart. Appl. Math. **15** (1957), 83–93.

The reflection and scattering of a plane acoustic wave impinging upon an elastic spherical shell and the corresponding stresses and displacemenrs in the shell are amalyzed through the use of Fourier transform techniques. Infinite series solutions in terms of spherical harmonics are found for the transforms of the fluid velocity potential and the shell displacement components. It is stated that the inversion has been carried out for the first three modes and some numerical results are presented for the case of a steel shell in water for an incident pressure pulse. Due to the physical discontinuities inherent in the problem, it is not practicable to obtain the acceleration of the shell and the pressure distribution on the shell from these series solutions. It is shown that the shell stress components can be obtained by means of the series solutions. However, from the graphical results presented, the reviewer would question the rapidity of the convergence or else raise the question of a mislabeling of scaling factors in the presentation of the results for the higher modes.

P. Chiarulli (Chicago, Ill.).

Zaid, Melvin; and Paul, Burton. Mechanics of high speed projectile perforation. J. Franklin Inst. **264** (1957), 117–126.

The perforation of a thin plate by a high-speed, conical-headed projectile is examined. The forces on the plate and projectile are deduced from momentum considerations, together with assumptions about the geometry of the deformation of the plate which are based on photographic evidence.

J. W. Craggs (Newcastle-on-Tyne).

Leonard, Robert W.; and Budiansky, Bernard. On traveling waves in beams. NACA Rep. no. **1173** (1954), iii+27 pp. (1955).

The equations of Timoshenko for vibrating nonuniform beams are used as a basis for the analysis of transient stresses. Three methods are employed; (a) the method of characteristics, which leads to a numerical step-by-step procedure, where a characteristic may be defined as a line across which discontinuities may exist; (b) the superposition of normal modes; (c) the derivation of exact solutions in closed form by means of the Laplace transform. In method (a) two sets of characteristics are obtained, corresponding to two propagation velocities c_1 and c_2. Discontinuities in the bending moment and in the rotational velocity of a cross-section are associated with one set; discontinuities in the shear and linear velocity of a cross-section with the other set.

The methods are illustrated by choosing examples for which exact solutions have been obtained, and the results compared with those given by methods (a) and (b) which are approximate in character. In these examples finite uniform beams, for which the propagation velocities c_1 and c_2 are equal, are considered; the variation with time of bending moment and shear is found for: (1) a cantilever beam given a step velocity at the root; (2) a simply

supported beam subjected to a step moment at one end; (3) a simply supported beam subjected to a ramp-platform moment at one end. The exact solutions are obtained for a semi-infinite beam, but the response of a finite simply supported beam is derived by superposing results for semi-infinite beams. Method (a) predicts large shear discontinuities when the discontinuous wave front returns after reflection at an end, and the existence of these discontinuities is confirmed by the exact solutions. Except in the regions immediately following the discontinuities, the predictions of the three methods show good agreement.
G. B. Warburton (Edinburgh).

Deresiewicz, H. Plane waves in a thermoelastic solid. J. Acoust. Soc. Amer. **29** (1957), 204–209.

The author considers wave propagation in an isotropic thermoelastic solid and shows that three types of wave may be propagated: (1) a purely elastic distortional wave; (2) a dilatational (E) wave which is modified by the thermal properties of the medium; (3) a thermal (T) wave which will be influenced by the dilatational properties of the medium. He shows that the E wave is propagated adiabatically at low frequencies and adiabatically at high frequencies. The phase velocity of the T wave increases monotonically with frequency. The specific loss is shown to reach a maximum for the E wave and a minimum for the T wave at the relaxation frequency for thermal currents. *H. Kolsky.*

Miklowitz, Julius. The propagation of compressional waves in a dispersive elastic rod. I. Results from the theory. J. Appl. Mech. **24** (1957), 231–239.

The author treats the propagation of extensional waves in a rod on the basis of approximate treatment put forward by Mindlin and Herrmann [Proc. 1st U.S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 187–191]. A numerical evaluation of the problem of a semi-infinite rod subjected to a step axial pressure is given. *H. Kolsky.*

Miklowitz, Julius; and Nisewanger, C. R. The propagation of compressional waves in a dispersive elastic rod. II. Experimental results and comparison with theory. J. Appl. Mech. **24** (1957), 240–244.

This part of the paper discusses experimental results. These are found to be in good agreement with the predictions of the theory as given in the paper reviewed above for the high frequency waves which travel more slowly down the rod. *H. Kolsky* (Providence, R.I.).

Skalak, Richard. Longitudinal impact of a semi-infinite circular elastic bar. J. Appl. Mech. **24** (1957), 59–64.

The author uses a double integral transform to investigate elastic wave propagation in a semi-infinite cylindrical bar and evaluates the solution for large times after the initial impact. The approximation involves an Airy integral and predicts dispersion of the wave immediately followed by damped oscillation. This is in agreement with observed experimental results. *H. Kolsky.*

Pursey, H. The launching and propagation of elastic waves in plates. Quart. J. Mech. Appl. Math. **10** (1957), 45–62.

Expressions are derived for the displacement of an arbitrary point in an isotropic elastic plate when a stress which varies sinusoidally with time is applied to the free surface. Two cases are considered. One is a two-dimensional field as in Lamb's treatment. The other is where there is circular symmetry about an axis normal to the surface of the plate. Tables are given of velocity of propagation as a function of wavelength in both symmetric and antisymmetric modes for Poisson's ratio of $\frac{1}{4}$ and $\frac{1}{3}$. Formulae are derived from which the amplitude and group velocity of a given wave may be calculated. *H. Kolsky.*

Hunter, S. C. Energy absorbed by elastic waves during impact. J. Mech. Phys. Solids **5** (1957), 162–171.

The expressions for the energy associated with the elastic waves generated by surface forces acting on a semi-infinite elastic medium are given, and, by means of Fourier synthesis, the energy loss for a Hertzian collision between a small sphere and a massive specimen is determined. The force-time profile is taken to be the positive half cycle of a sinusoid and it is shown that at low velocities of impact only a small fraction ($<1\%$) of the kinetic energy of the sphere is converted into vibrational energy of the specimen. It is concluded that the higher loss observed experimentally on steel specimens must be associated with some form of internal friction.
H. Kolsky (Providence, R.I.).

Lessen, Martin. Thermoelastic damping at the boundary between dissimilar solids. J. Appl. Phys. **28** (1957), 364–366.

The author first derives thermoelastic equations for an initially stressed isotropic solid. The propagation of a longitudinal disturbance across the boundary between two dissimilar solids for the case where wavelength is much larger than thickness of thermal boundary layer is considered. It is shown that under these conditions the energy loss per unit time due to thermoelastic damping is proportional to the square root of the disturbance frequency. *H. Kolsky* (Providence, R.I.).

Chakraborty, S. K. On disturbances produced in an elastic medium by twists applied on the inner surface of a spherical cavity. Geofis. Pura Appl. **33** (1956), 17–22.

The author considers the disturbance produced by uniform torque applied to the inner surface of a spherical cavity. Deformation is here purely distortional and the amplitude of the outgoing wave is calculated for the stress decaying exponentially with time and for damped oscillatory motion. *H. Kolsky* (Providence, R.I.).

Chakraborty, S. K. Disturbances of cylindrical origin in an isotropic elastic medium. Geofis. Pura Appl. **33** (1956), 9–16.

The author considers disturbances produced by a source acting over a small region of the inner surface of a cylindrical hole. The decay of amplitude with distance from the source is calculated for unbounded solids and for infinite slabs. *H. Kolsky* (Providence, R.I.).

Malyutin, I. S. On equilibrium of plates under pressure beyond the elastic limit. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 5, 118–121. (Russian)

A rectangular plate is compressed by equal uniformly distributed forces on the two long sides. It is assumed that deformation takes place only in a narrow strip parallel to and midway between the long sides. Considering the problem from the point of view of, first, the theory of small elasto-plastic deformation and, second, the theory of flow, the author obtains implicit relations

between the deflection and the external load. He discusses properties of these relations, such as the load range in which bifurcation can occur. A graphical comparison of the results of the two theories for a particular case is displayed. *R. N. Goss* (San Diego, Calif.).

Nowiński, Jerzy. States of stress in an orthotropic thick-walled tube surrounded by an elastic medium. The Galerkin problem for an orthotropic material. Arch. Mech. Stos. **9** (1957), 217–226. (Polish and Russian summaries)

The paper is concerned with the states of stress and deformation in an orthotropic thick-walled elastic tube. It is assumed that the principal directions of elastic and thermal orthotropy coincide with the radical and circumferential directions. Under this assumption, the problem reduces to the solution of a simple linear ordinary differential equation. *E. T. Onat* (Providence, R.I.).

Mansfield, E. H. Studies in collapse analysis of rigid-plastic plates with a square yield diagram. Proc. Roy. Soc. London. Ser. A. **241** (1957), 311–338.

The paper is concerned with the determination of upper bounds to the collapse loads of a rigid-plastic plate with a square yield diagram. As in all fracture line theories, in the present analysis attention is confined to mechanisms of collapse in which one of the principal curvatures remains zero. Previous Russian work [A. R. Rzhanitsyn, Structural analysis taking account of plastic properties of materials, 2nd ed., Gos. Izdat. Lit. Stroitel. Arhitek., Moscow, 1954, Ch. VIII], contains many of the results presented in this paper. However, the present paper was conceived independently and in many respects it is more general. *E. T. Onat* (Providence, R.I.).

★**Drucker, D. C. A definition of stable inelastic material.** Tech. Rep. No. 2, Nonr–562 (20), Division of Engineering, Brown University, Providence, R. I., September, 1957. 23 pp.

Broad definitions are given for stability in work-hardening and perfect plasticity including viscous effects. The system considered includes both body and time-dependent forces acting on it. An external agency is supposed to apply an additional set of forces and it is postulated that the work done by the external agency on the additional displacements it produces is positive. Some of the restrictions thus imposed on permissible stress-strain relations are explored. Uniqueness of solution is studied. (From the author's summary.)

E. H. Mansfield (Farnborough).

Hill, R. On the problem of uniqueness in the theory of a rigid-plastic solid. III. J. Mech. Phys. Solids **5** (1957), 153–161.

This is the third of a series of papers by the author on uniqueness theory for a rigid-plastic solid [see same J. **4** (1956), 247–255; **5** (1956), 1–8; MR **18**, 83, 431]. The work differs from the earlier papers in that changes in geometry are included. The integral which forms the basis for the analysis thus has an additional term. A sufficient condition for uniqueness is obtained. By considering the relation between the rate of hardening and the uniqueness criterion, the author is able to distinguish cases where the indeterminacy for a non-hardening solid can be removed by a limiting process. The uniqueness criterion is investigated for the special cases of uniaxial tension, compression of a strut, and torsion. In the last case, it is shown that neglecting positional changes will yield the correct mode but not necessarily the correct distribution of stress-rate. The effect of uniform normal pressure is also considered. *G. H. Handelman.*

Olszak, Wacław; Perzyna, Piotr; and Szymański, Czesław. Two-dimensional problems in the theory of plasticity of non-homogeneous anisotropic bodies. Arch. Mech. Stos. **9** (1957), 335–358. (Polish and Russian summaries)

The authors start with a general three dimensional formulation of the governing equations for a perfectly plastic anisotropic material. The stress-strain rate law is based on the quadratic plastic potential with arbitrary coefficients.

Two dimensional problems are first categorized as two dimensional strain or stress problems. The most general two dimensional strain problem assumes only that $d\varepsilon_{ij}=d\varepsilon_{ij}(x_1, x_2)$ with no restrictions on the resulting stress tensor. Next restriction is to set $d\varepsilon_{33}=0$; authors term this a "generalized plane state of strain." Next a "plane state of strain" requires $d\varepsilon_{i3}=d\varepsilon_{3i}=0$, but still no restriction the stress state. Finally, the "classical plane state of strain" is obtained by restricting the coefficients of the plastic potential so that $\sigma_{13}=\sigma_{23}=\sigma_{31}=\sigma_{32}=0$. The governing equations for these four degrees of generality are written in a form which allows the plastic moduli to vary from point to point.

Next, various special cases of anisotropy (monoclinic, orthogonal, transverse, cubic, and isotropy) are investigated. It is shown that further restrictions on the plastic moduli are implied for each special case. Finally, a similar treatment is given for the corresponding generalized plane stress problems. *P. G. Hodge, Jr.* (Chicago, Ill.).

Satoh, Tunezo. Two dimensional plastic flow. III. Mem. School Sci. Engrg. Waseda Univ., Tokyo no. 20 (1957), 43–49.

This paper concerns stress distributions in perfectly-plastic rigid material under conditions of plane plastic strain. It is well-known that such stress states may be specified simply in terms of two non-dimensional dependent variables ω and θ, themselves functions, say, of rectangular Cartesian co-ordinates x and y [see, e.g., W. Prager and P. G. Hodge, Jr., Theory of perfectly plastic solids, Wiley, New York, 1951, § 21; MR **14**, 430]. Substitution into the equilibrium equations furnishes two simultaneous, non-linear, partial differential equations for ω and θ. Here, following S. Hristianovič [Mat. Sb. N.S. **1** (43), (1936), 511–534], the author interchanges the rôles of the dependent and independent variables in these equations to furnish two simultaneous, linear, partial differential equations for x and y now regarded as functions of ω and θ. The main object is to show that these (real) equations may be simply combined together in a single complex equation, and thence the solution is expressed formally in terms of a function satisfying a certain hyperbolic partial differential equation. The author has in mind, principally, an inverse method of attack on the problems of plane plastic strain. By way of illustration, the solution is obtained, for example, of Prandtl's problem of a long rectangular block compressed between two rough parallel rigid plates. {Reviewer's remark: Previous experience [see, e.g., I. N. Sneddon, Phil. Mag. (7) **36** (1945), 629–635; MR **8**, 115] of semi-inverse methods of approach to plasticity problems suggests that, although these methods are of general

interest, they are nevertheless of limited use in the solution of outstanding problems of practical interest.}

H. G. Hopkins (Sevenoaks).

Ericksen, J. L. Characteristic directions of certain equations of plasticity. Mathematika 3 (1956), 56–62.

The author considers an incompressible, ideally plastic material defined by the stress-deformation relations

$$s_{ij}=F_1e_{ij}+F_2(e_{ik}e_{kj}-\tfrac{1}{3}\delta_{ij}), \tag{1}$$

where s_{ij} is the deviatoric stress tensor and

$$e_{ij}=(d_{km}d_{km})^{-\frac{1}{2}}d_{ij}$$

where d_{ij} is the rate of deformation tensor. F's are assumed to be continuously differentiable functions of $\omega=3\times 6^{\frac{1}{2}}\det e_{ij}$. To avoid complex variables it is assumed that $-1\leqq\omega\leqq 1$. Following a previous paper [M. Baker and J. L. Ericksen, J. Washington Acad. Sci. **44** (1954), 33–35; MR **16**, 89], it is further assumed that

$$F_1>F_2e_M \text{ if } e_N\neq e_P \quad (M, N, P \text{ distinct}),$$
$$F_1\geqq F_2e_M \text{ if } e_N=e_P$$

where e_N are the eigenvalues of e_{ij}. Characteristic directions for the system of equations comprising equations of motion, the condition of incompressibility and the stress-deformation relations (1) are then investigated.

It is first shown that if $\omega=0$ (one eigenvalue of e_{ij} is zero) real characteristic directions exist and these make the angle $\varphi=\pm\pi/4$ with the eigenvectors corresponding to non-zero eigenvalues.

The case of $\omega=\pm 1$ (e_{ij} has equal eigenvalues as in the case of simple extension) is considered next. It is shown that there exist two families of real characteristics if and only if $F_1(\pm 1)\mp 2F_2(\pm 1)/\sqrt{6}=0$.

Finally, the author remarks that it would be of interest to determine necessary and sufficient conditions on the F's for there to exist real characteristic directions for all ω in the interval $-1\leqq\omega\leqq 1$. *E. T. Onat.*

Parsons, D. H. Plastic flow with axial symmetry: A note on the effect of different flow criteria. J. London Math. Soc. 32 (1957), 233.

It is shown that some of the results of an earlier paper of the author's [Proc. London Math. Soc. (3) **6** (1956), 610–625; MR **18**, 690] are not restricted to the von Mises flow criterion there used, but can be generalised for any flow criterion. *W. F. Freiberger.*

Freiberger, Walter. Minimum weight design of cylindrical shells. J. Appl. Mech. 23 (1956), 576–580.

The paper is concerned with the development of a method for designing the variable wall thickness of a cylindrical shell under axial loading and arbitrary pressures to give maximum economy of material. The shell is composed of an ideally plastic material and following a previous analysis [E. T. Onat and W. Prager, De Ingenieur **67** (1955), *O.* 46–*O.* 49] it is assumed that for maximum economy of material the generic point of the stress state of the shell must move on the edges separating the various smooth domains of the yield surface defined in terms of generalized stresses. Using this assumption and the equations of equilibrium the optimum thickness distribution is obtained for a variety of loading conditions.

As the author remarks in a later paper [Jour. Aero. Sci. **24**, pp. 847–848)] the criterion of minimum weight design used in the present paper has its shortcomings and is open to objection. (For instance, it is not applicable to entirely smooth yield surfaces.) *E. T. Onat.*

Barta, J. On the minimum weight of certain redundant structures. Acta Tech. Acad. Sci. Hungar. **18** (1957), 67–76. (German, French and Russian summaries)

The author establishes that for statically indeterminate structures, the minimum weight design occurs when sufficient constraints (i.e., redundancies) are removed so that structure becomes statically determinate. The resulting design may not be unique, but the weight is an absolute minimum. For example, for a redundant truss, sufficient bars are removed to make the truss statically determinate, but there may be several ways of doing this, leading to different designs all having the same (minimum) weight.

The principle has been known for some time, but does not appear to have been discussed in the literature; Sved gave a restricted proof for an idealized stress-strain curve [Austral. J. Appl. Sci. **5** (1954), 1–9], but the present proof is valid for any (reasonable) stress-strain law.

J. Heyman (Providence, R.I.).

Il'yušin, A. A. Complete plasticity of processes of flow between rigid surfaces, analogy with a sandy embankment and some applications. Prikl. Mat. Meh. **19** (1955), 693–713. (Russian)

It is supposed that the basic conditions of complete plastic flow between rigid surfaces are that the stresses are defined by the following relations: $\sigma_{11}\approx\sigma_{22}\approx -q$, $\sigma_{33}=-p\approx-(q+\sigma_s)$, $\sigma_{12}\approx 0$. The kinematical conditions show that transverse velocities are constant. These two kinds of conditions represent natural hypotheses by means of which the equations of the first basic problem can be deduced. Using one basic surface and supposing that it is immovable, the author deduces the equations of interior motion. Then the case when the interior motions of both rigid bodies are equal is treated, and an analogy with a sandy embankment is given. The theory is applied to several cases (rectangular plates with parallel, crossed or other kinds of stringers). Also, the rolling problem is treated, assuming that the contact length is sufficiently greater than the height of the material at the beginning of rolling. *D. Rašković.*

Johnson, W.; and Mellor, P. B. The centre of shear for a material having a non-linear stress-strain curve. Appl. Sci. Res. A. **6** (1957), 467–477.

A uniaxial stress-strain relation of the form $p=A\varepsilon^n$ is used together with the simple engineering theory of bending to determine the shear center of a thin-walled cantilever. Under these assumptions Phillips [Introduction to plasticity, Ronald Press, New York, 1956, p. 53] has given a simple bending law. The authors then derive expressions for the shear-center of a semi-circular section and a channel section which are particularly simple. The results for the channel section agree well with a range of values found by the reviewer [J. Aero. Sci. **18** (1951), 749–754, 766; MR **13**, 603] using a more general stress-strain curve. Under these assumptions, the authors show that the shear center is independent of the applied load. *G. H. Handelman* (Troy, N.Y.).

★ **Арутюнян, Н. Х. [Arutyunyan, N. H.] Некоторые вопросы теории ползучести. [Some questions of the theory of creep.]** Gosudarstv.Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 323 pp. 10.70 rubles.

The content of this book is divided into three parts with eight chapters. The first part deals with the basic equations of creep theory. The second part presents

temperature stresses and deformations in concrete, including creep of the material. Also, statically indeterminate systems are treated. The last part deals with reinforced concrete constructions. Explanations are very clear. Quoted literature is predominantly Russian.

D. Rašković (Belgrade).

Sanders, J. Lyell, Jr.; McComb, Harvey G., Jr.; and Schlechte, Floyd R. A variational theorem for creep with applications to plates and columns. NACA Tech. Note no. 4003 (1957), 23 pp.

An analog of Reissner's variational theorem in elasticity is formulated for a body undergoing elastic, plastic, and creep strains. Finite slopes and deflections are taken into account but strains are assumed infinitesimal. The creep strain is supposed to depend upon the stress deviator but not on the stress rate nor the hydrostatic stress. The integral to be varied was found by trial and error guided by the extremum theorems of Wang and Prager. Applications are made to creep problems only. In the illustrative problems, linear distribution of both stress and strain through the thickness are substituted in the integral.

D. C. Drucker (Providence, R.I.).

Narasimhamurthy, P. An analytical investigation of creep under combined loadings. Aircraft Engrg. 29 (1957), 346–349.

This paper deals with the analytical formulation of the laws of creep under combined loading, with extension to the anisotropic case and experimental verification. The problem is important in aero-dynamic heating during supersonic flight.

Litwiniszyn, Jerzy. Application of the equation of stochastic processes to mechanics of loose bodies. Arch. Mech. Stos. 8 (1956), 393–411.

The model of a soil analysed stochastically in this paper is that of a uniform aggregate of loose particles, piled up in such a way that removal of one particle at a lower level will cause a disturbance in the upper layers. In the x, z-plane, where z is the vertical direction, particles originally located on the line $z=z_1$ will after displacements occupy positions on the curve $z=z_1-w(x, z_1)$; the displacement at (x_2, z_2) due to displacements along the line $z=z_1$ is expressed by

$$w(x_2, z_2)=\int_{-\infty}^{\infty}\varphi(x, z_1; x_2, z_2)w(x, z)dx,$$

where the distribution function φ is shown to satisfy the ordinary Einstein-Smoluchowski integral equation

$$\varphi(x, z_1; \eta, z_3)=\int_{-\infty}^{\infty}\varphi(x, z_1; \xi, z_2)\varphi(\xi, z_2; \eta, z_3)d\xi.$$

It is further assumed that

$$\int_{-\infty}^{\infty}\varphi(x, z; \xi, z+\Delta z)d\xi=1-\frac{\partial\alpha(z, t)}{\partial z}\Delta z,$$

where t is the time. This assumption means that (1) $\partial\alpha/\partial z=0$ characterizes an incompressible medium, (2) the removal with probability 1 of a particle at (x, z) results in an effect with probability $1-\partial\alpha/\partial z\cdot\Delta z$ at level $z+\Delta z$. This probability is generally assumed directly proportional to the time-delay after the removal of the particle at level z.

On the basis of these assumptions it is shown that $\varphi(x, s; \eta, z)$ satisfies

$$\frac{\partial\varphi}{\partial z}=-\varphi\frac{\partial\alpha}{\partial z}-\frac{\partial(\varphi A)}{\partial\eta}+\frac{\partial^2(\varphi\cdot B)}{\partial\eta^2}$$

and that $w(x, z)$, determining the depth-coordinates of a settlement basin, satisfies

$$\frac{\partial w}{\partial z}=w\frac{\partial\alpha}{\partial z}-A\frac{\partial w}{\partial x}-B\frac{\partial^2 w}{\partial x^2},$$

where $A=A(x, z)$, $B=B(x, z)$ are functional coefficients.

The development is generalised to three dimensions; it represents an interesting new application of a well-known formalism. Reference is made to experimental checks.

W. F. Freiberger (Providence, R.I.).

Grigoryan, S. S. On axi-symmetric motions of a friable material. Prikl. Mat. Meh. 21 (1957), 221–230. (Russian)

Nowiński, Jerzy. The principle of stationary free energy in the thermoelastic analysis of thin-walled tubes. Arch. Mech. Stos. 9 (1957), 359–368. (Polish and Russian summaries)

The author considers thin-walled tubes with an arbitrary steady temperature distribution and subject to arbitrary steady external loads producing shear, bending, and torsion. Within the usual approximations of the linear theory he derives the differential equations and the boundary conditions for the deformation of the tube from a variational principle. This principle states that the total free energy is a minimum in the state of equilibrium.

W. Noll (Pittsburgh, Pa.).

Chowdhury, Pritindu. Two-dimensional thermal stresses due to periodic supply of heat on the straight edge of a semi-infinite plate. J. Assoc. Appl. Phys. 4 (1957), 1–6.

The thermal displacement, expressed as grad ψ, is related to the temperature T by $\partial\psi/\partial T=(1+\nu)k\alpha T$, the constants ν, k, α having their usual meanings. Combined with $k\nabla^2 T=\partial T/\partial t$, the thermal stresses in a semi-infinite isotropic plate produced by a flame on the straight boundary, are expressed as a Fourier integral.

B. R. Seth (Kharagpur).

Nowacki, Witold. A three-dimensional thermoelastic problem with discontinuous boundary conditions. Arch. Mech. Stos. 9 (1957), 319–324. (Polish and Russian summaries)

The stationary state of stress is analyzed in the elastic halfspace with prescribed constant temperature over a circle of given radius within the free boundary plane, the rest of this plane being thermally insulated. The complete solution is obtained by superimposing upon a particular solution of the well-known potential equation for the thermo-elastic strain function a solution of the biharmonic equation for the isothermal stress function which ensures the stress-free boundary.

A. M. Freudenthal (New York, N.Y.).

Nowacki, Witold. A dynamical problem of thermoelasticity. Arch. Mech. Stos. 9 (1957), 325–334.

The state of stress in the infinite elastic continuum due to a concentrated heat-source is analyzed under the conditions of (a) instantaneous application of the source, represented by a Dirac-function, (b) continuous heat-source of constant intensity and, (c) heat-source of periodically varying intensity, considering the inertial term in the single differential equation for the radial displacement component. The solutions of the equation of the thermo-

elastic strain function for the three conditions are obtained with the aid of Laplace and Fourier-Hankel transforms respectively. *A. M. Freudenthal* (New York, N.Y.).

Kochanski, S. L.; and Argyris, J. H. Some effects of kinetic heating on the stiffness of thin wings. A preliminary investigation of the effects of thermal stresses on the torsional and flexural stiffness of thin solid wings. Aircraft Engrg. **29** (1957), 310–318.

This paper presents a preliminary analysis of the effects of kinetic heating at supersonic speeds on the torsional and flexural stiffnesses of thin solid wings. The main investigation is based on the small deflexion theory, but the scope of the analysis for torsion is extended to cover the effects of large deformations. *Authors' summary.*

See also: Differential Geometry: Lagally. **Structure of Matter:** Laval. **Geophysics:** Malinovskaya.

Structure of Matter

Mueller, Charles R.; and Stupegia, Donald C. Lattice vacancy theory of the liquid state. J. Chem. Phys. **26** (1957), 1522–1525.

Laval, Jean. L'Elasticité du milieu cristallin. II. Dynamique des ondes élastiques. J. Phys. Radium (8) **18** (1957), 289–296.

The fundamental assumption of central forces in the classical theory of elasticity does not allow it to explain the different velocities of propagation along the axes of symmetry of a polyatomic crystal. Continuing his work on the elasticity of crystalline media [same J. (8) **18** (1957), 247–259; MR **19**, 199] the author analyses wave propagation with the help of a Hermitian Fourier matrix. Polarisation in opposite senses caused by elastic waves and the deformation of the ions accounts for interatomic forces. It vanishes in ordinary elastic bodies.

For low frequencies, which correspond to waves in elastic media, results are deduced and compared with those given by the classical theory. The classical matrix A of the elastic constants is related to the covariant displacement vector and to the contravariant force constants, but in the case of the atomic matrix B this order is reversed. When the forces are central A and B are comformable, and the two theories give similar results for low frequencies. But in general, for vibrations in polyatomic crystals the matrix B, and not the matrix A, gives results in agreement with experiment. *B. R. Seth.*

Laval, Jean. L'Elasticité du milieu cristallin. III. Les déformations linéaires (élasticité statique). J. Phys. Radium (8) **18** (1957), 369–379.

In this third paper [for part II, see the paper reviewed above], the author defines linear or homogeneous deformation as one which conserves the triple periodicity of the crystalline medium. For a polyatomic crystal it is shown that the deformation tensor need not be symmetric. Two kinds of deformations are obtained — one kinetic and the other static. These cannot be related to the same coefficients of elasticity and can give rise to an internal couple even when no electric or magnetic forces are present.

In general the corresponding strain energy is found to have components of rotation as well. For low frequencies the distinction between the two coefficients of elasticity disappears only if the forces are central. All the results of the classical theory can then be deduced.

The object of all these studies is to show that the classical theory of elasticity is essentially based on the idea of central forces. This condition is satisfied by all monoatomic crystals. But, in general, polyatomic crystals do not obey this law. The consequences are that internal moments are produced, the deformation tensor becomes asymmetric, and the velocities of wave propagation are not the same along the axes of symmetry of the crystal. In such cases the results of the classical theory of elasticity cannot be used. *B. R. Seth* (Kharagpur).

Yoshioka, Hide. Effect of inelastic waves on electron diffraction. J. Phys. Soc. Japan **12** (1957), 618–628.

The effect of inelastic scattering on electron diffraction by crystals is theoretically studied. It is proved that the effect of inelastic scattering gives rise to an additional complex-term Fourier coefficient of the inner potential. The imaginary part of the additional term explains qualitatively the observation that two components of a doubly refracted electron beam are subjected to different degrees of absorption [Honjo and Mihama, same J. **9** (1954), 184–198]. The real part of the additional term gives a correction of about 6–7% to the calculated value of mean inner potential at accelerating voltages of 30–40 kv. This explains the systematic difference between the mean inner potential experimentally obtained by electron diffraction and that calculated by the use of diamagnetic susceptibility [Miyake, Proc. Phys.-Math. Soc. Japan (3) **22** (1940), 666–676]. *W. Nowacki* (Bern).

Yomosa, Sigeo; and Nagamiya, Takeo. Diffuse scattering of X-rays by a point-imperfection in diamond lattice. J. Phys. Soc. Japan **12** (1957), 610–617.

It is assumed that when a carbon atom is removed from the diamond structure, or replaced by a foreign atom of tetrahedral valency four, the neighbouring carbon atoms are displaced with cubic symmetry by amounts calculable from the elastic constants. From the Fourier transforms of the displacements the resultant intensity of diffuse scattering is calculated as a function of position in reciprocal space. Qualitatively similar results are obtained for nearest-neighbour interactions only and for next-nearest-neighbour interactions also. For 111 reflexions there are regions extending in [111] directions, 113 reflexions in roughly [001] directions, and for 220 reflexions in [110] directions. This pattern does not resemble the spikes in [100] directions observed with type-I natural diamonds, but may have applications to radiation-damaged diamonds.

The method is based on that of Matsubara [same J. **7** (1952), 270–274]. *A. J. C. Wilson* (Cardiff).

Belov, N. V.; and Tarhova, T. N. Nomographic method of computing structure factors. Akad. Nauk SSSR. Kristallografiya **1** (1956), 235–238. (Russian)

A nomographic method of computing structure factors for X-ray determination of crystal structures is described, applicable for space groups when the structure factor is written as

$$A = \cos 2\pi(hx+ky) + \cos 2\pi(hx-ky)$$

or a similar expression.

The multiples of values of x and y are plotted on the right and left edges of the nomogram, respectively, and labelled with the corresponding integers h and k. Line

joining points h and k intersects a cosine scale at the value which represents contribution of a single atom to the $(hk0)$ structure factor. *V. Vand* (University Park, Pa.).

Cochran, W.; and Kartha, G. Scattering of X-rays by defect structures. II. An extension of the theory. Acta Cryst. **9** (1956), 941–943.

The theory of [Cochran, Acta Cryst. **9** (1956), 259–262; MR **19**, 85] the X-ray scattering of crystals containing defects described earlier is extended to cover the cases where displacements of lattice points may be large, and brought about by the simultaneous operation of a number of defects.

It is assumed that the components of the displacement of an atom in the direction of the reciprocal lattice vectors have a Gaussian distribution. An expression for the intensity of the reflection produced from a crystal containing defects is then derived and shown to be identical with that obtained from the theory of thermal diffuse scattering [R. N. James, the optical principles of the diffraction of X-rays, Bell, London, 1948].

Finally, it is shown that the diffuse intensity can be expressed in terms of the Fourier transforms of the individual defects. *W. M. Macintyre* (Boulder, Colo.).

Cochran, W.; and Kartha, G. Scattering of X-rays by defect structures. III. The effect of interstitial atoms and vacancies. Acta Cryst. **9** (1956), 944–948.

The case of a crystal containing a random distribution of vacancies and interstitial atoms is considered. Expressions are derived for the intensity of the diffuse scattering from the crystal and also for the change in the Laue - Bragg intensity. The specific case of defects in a copper crystal is discussed, and the iso-diffusion contours around selected reciprocal lattice points are investigated. It is shown that the influence of interstitial atoms on the diffuse intensity and on the Bragg-Laue intensity is more important than that of vacancies.

Modifications required in the theory as a result of a non-random distribution of defects are also discussed.

W. M. Macintyre (Boulder, Colo.).

Haug, Albert; und Schönhofer, Alfred. Energiebandaufspaltungen und Zwischenbandterme bei Verschiebung von Gitteratomen im Festkörper. Z. Physik **148** (1957), 513–526.

An Hand eines eindimensionalen Modells mit δ-artigen Potentialen wird die Modifizierung des Energiespektrums eines Elektrons im Kristall infolge der Auslenkungen der Gitterbausteine aus ihren Gleichgewichtslagen untersucht, wobei eine Auslenkung wirklich durch eine Verschiebung des δ-Potentials dargestellt wird. Danach bewirken die von den Gitterschwingungen herrührenden, räumlich periodischen Auslenkungen sowohl eine Aufspaltung der ursprünglichen Energiebänder wie auch eine Verschiebung ihrer Grenzen, so dass unter Umständen eine Ueberschneidung der Bänder auftritt. Dagegen treten bei der Auslenkung eines einzelnen Gitterteilchens in jeder Energielücke zwei Zwischenbandterme zusätzlich zu den Bändern des ungestörten Problems auf.

W. Nowacki (Bern).

Schlömann, Ernst. Theory of infra-red resonances in ferrimagnetics. Phys. and Chem. Solids **2** (1957), 214–220.

The frequency dependence of the magnetic susceptibility for circular excitation of an uncompensated ferrimagnet is derived from the Kramers-Kronig relations for gyrotropic media. In the lossless case the susceptibility is completely determined by two sets of parameters: the resonance frequencies and the corresponding optical strengths. *From the author's summary.*

★ **Laval, J. Diffusion des rayons X provoquée par l'agitation thermique des atomes dans les cristaux.** International conference on current problems in crystal physics. pp. 267–276. Massachusetts Institute of Technology, Cambridge, Mass., July 1–5, 1957.

Der Autor gibt einen kurzen zusammenfassenden Ueberblick über experimentelle Methode und Theorie. Vgl.: Bull. Soc. Franç. Minéral. **62** (1939), 137–253; **64** (1941), 1–138; J. Phys. Radium (8) **15** (1954), 545–558, 657–666. *W. Nowacki* (Bern).

★ **Адирович, Э. И. [Adirovič, È. I.] Некоторые вопросы теории люминесценции кристаллов. [Some questions in the theory of luminescence of crystals.]** 2nd ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 350 pp. 10.45 rubles.

Fluid Mechanics, Accoustics

★ **Tietjens, O. G. Applied hydro- and aeromechanics, based on lectures of L. Prandtl; translated by J. P. Den Hartog.** Dover Publications, Inc., New York, N. Y., 1957. xvi+311 pp. $1.85.

Reprint of the volume first published in 1934. It treats, principally, of flow in pipes and channels, boundary layers, drag of bodies moving through fluids, airfoil theory and additional material by O. G. Tietjens on experimental methods.

Bjørgum, Oddvar. On the physical boundary conditions in fluid dynamics. Univ. Bergen Årbok. Naturvit. Rekke **1955**, no. 4, 8 pp. (1956).

By a method of power series expansion of the velocity, the author considers the restrictions imposed by the boundary conditions appropriate to fixed boundaries with no slip and with skin friction. The results serve to exclude certain types of vector fields as admissible solutions to hydrodynamical problems. *C. Truesdell.*

Proskuryakov, A. P. Theory of rotor at zero angle of attack. Prikl. Mat. Meh. **20** (1956), 519–531. (Russian)

Expanded version of report presented to an aerodynamics seminar of the Mechanics Institute, AN USSR in 1953 considers flow over a rotor at zero angle of attack in which the formulas for the induced velocity are derived for all space. An analysis is also made of the vortex sheets within the rotor disc and beyond the rotor. The representations of the flow from the rotor as a layer of longitudinal rectilinear vortices and a layer of circular vortices or as a layer of trochoidal vortices are shown to give equivalent results. *M. D. Friedman.*

Heaslet, Max. A. The minimization of wave drag for wings and bodies with given base area or volume. NACA Tech. Note no. **3289** (1957), 28 pp.

Morikawa, G. K. Non-linear diffusion of flood waves in rivers. Comm. Pure Appl. Math. **10** (1957), 291–303.

When a resistance term is included, the hydraulic

equations for flow in an open channel of constant slope can describe the so-called 'steady progressing wave' which moves downstream with a steady profile. The speed U of this wave is determined by the asymptotic values h_0 and h_1 of the downstream and upstream depths. The author uses expansions of the flow quantities in powers of $\lambda=(h_1-h_0)/h_0$ to study how neighboring profiles approach the steady progressing wave as $t\to\infty$. For this purpose he introduces a stretching transformation in which the flow quantities are taken to be functions of the independent variables $\omega=\lambda(x-Ut)$, $\tau=\lambda^2 t$. The lowest order equation is then found to be Burgers's equation. The deviations from the steady profile wave damp out exponentially.

The author's comments on the relation of this work to the more general 'kinematic wave theory' of floods [M. J. Lighthill and G. B. Whitham, Proc. Roy. Soc. London. Ser. A. **229** (1955), 281–316; MR **17**, 309] are unsound, since he confuses 'kinematic shocks' (which are in fact the steady progressing waves) with 'dynamic shocks' (which are bores). *G. B. Whitham.*

Crane, L. J.; and Pack, D. C. The laminar and turbulent mixing of jets of compressible fluid. I. Flow far from the orifice. J. Fluid Mech. **2** (1957), 449–455.

This paper deals with the problem of laminar and turbulent mixing of jets of compressible fluids. By assuming a coefficient of eddy kinematic viscosity, the equations of motion for both cases are combined into one and can be solved by iteration. In the case where the Prandtl number is unity and viscosity varies with the n-th power of temperature, the second order solution is expressible in terms of Legendre polynomials.

Y. H. Kuo (Peking).

Keith, H. D. Simplified theory of ship waves. Amer. J. Phys. **25** (1957), 466–474.

The physical significance of group velocity is considered critically and a general and realistic definition given. This is used to develop a simple phenomenological theory of the ship wave pattern including the prediction of phase changes at singularities, and the variation of the wave-making resistance of a ship with speed is shown to be governed in a simple way by the interference of bow and stern waves. *Author's summary.*

Bjørgum, Oddvar. On the differential equations for the velocity field in flows of fairly general fluids. Univ. Bergen Årbok. Naturvit. Rekke **1955**, no. 3, 20 pp. (1956).

The author considers motion of a perfect gas obeying the Navier-Stokes equations and the Stokes relation. He obtains necessary and sufficient conditions to be satisfied by the velocity field $\mathbf{v}$ in order that pressure, density, and temperature fields exist such as to render this velocity a possible one. Set

$$\mathbf{U}\equiv\dot{\mathbf{v}}-\mathbf{F}-\tfrac{1}{3}\nu\nabla\nabla\cdot\mathbf{v}-\nu\nabla^2\mathbf{v},$$

where $\mathbf{F}$ is the extrinsic force, assumed to be a given function of position and of $\mathbf{v}$, and ν is a constant. For steady baroclinic motion, elimination of pressure and density yields

$$\mathbf{U}\cdot\nabla\times\mathbf{U}=0,$$

$$\nabla\times\left\{\frac{\mathbf{v}\times\nabla\times\mathbf{U}}{\mathbf{v}\cdot\mathbf{U}}+\frac{\nabla\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{U}}\mathbf{U}\right\}=0;$$

for steady barotropic motion,

$$\nabla\times\mathbf{U}=0,$$

$$\mathbf{U}\times\nabla\left(\frac{\nabla\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{U}}\right)=0.$$

In these preliminary results, the energy equation and the equation of state are not regarded. The author goes on to consider the full set of equations in possibly unsteady motion with prescribed heat supply. Various cases are distinguished. For baroclinic motion, ten scalar equations to be satisfied by $\mathbf{v}$ result. The details of this complicated but clearly explained analysis cannot be reported in this review. *C. Truesdell.*

Philippova, L. A. Unsteady motion of a viscous incompressible fluid in a narrow slit of constant breadth in the presence of heat transfer. Vestnik Leningrad. Univ. **12** (1957), no. 1, 141–151, 210–211. (Russian. English summary)

A problem in viscous flow with heat conduction and energy dissipation under conditions where convective effects may be neglected is examined under the assumption of constant fluid properties, including density. The pressure gradient in the narrow two-dimensional slot is assumed to be time and position dependent. Heat is transferred to the walls by conduction. One wall is assumed to be stationary with the other moving. Since the slot is narrow the results are applicable to annular regions as in the hydrodynamic theory of lubrication. Explicit formulations are given for the velocity and temperature distribution for arbitrary boundary conditions. *N. A. Hall* (New Haven, Conn.).

Moore, D. W. The flow past a rapidly rotating circular cylinder in a uniform stream. J. Fluid Mech. **2** (1957), 541–550.

A circular cylinder is rotating in a uniform stream of viscous liquid, the circumferential velocity of the cylinder being much greater than that of the uniform stream. It is shown that the effect of the uniform stream can be regarded as a small perturbation of the rotary flow due to the cylinder, and that a uniformly valid first approximation to the flow field can be obtained in this way. This approximate solution has the interesting property that the circulation about any circular contour concentric with the cylinder is the same. The lift on the cylinder is shown to be that given by the classical formula, and the drag force is zero to the order of the approximation considered. *Author's summary.*

★ **Coles, Donald. The laminar boundary layer near a sonic throat.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 119–137. Stanford University Press, Stanford, Calif. $7.50.

This paper contains boundary layer calculations for a convergent-divergent channel with sonic velocity of the throat. The following assumptions are made – Prandtl number unity, viscosity proportional to temperature, constant wall temperature, laminar profile of Falkner-Skan type with large values of the parameter $\beta=2u/u+1$. The Howarth-Stewartson transformation is used and the problem is technical to the solution of a pair of ordinary differential equations. A singular perturbation method is then introduced involving dependence of both dependent and independent variables on the parameter β. More precisely, the former are supposed to be inverse power

series of $\sqrt{\beta}$. Separate inner and outer expansions are required. Numerical calculations for the first approximation are compared with the results of Cohen and Reshotko [NSCS Tech. Note no. 3325 (1955); MR 16, 876].

A. Robinson (Jerusalem).

★**Baron, Judson R. The heterogeneous, laminar, boundary layer.** Tech. Rep. 236, presented at the symposium for mass-transfer cooling for hypersonic flight. Massachusetts Institute of Technology, Naval Supersonic Laboratory, Cambridge, Mass., 1957. iii+27 pp.

The results of a previous report by the author [Tech. Rep. 160, Mass. Inst. Tech., Naval Supersonic Lab., 1956; MR 18, 531] are summarized and a few new results are obtained. In particular, a similarity analysis is presented for boundary layers with external pressure gradients, and a brief study is made of the heterogeneous boundary layer in the case of a sublimating solid surface.

D. W. Dunn (Ottawa, Ont.).

Becker, E. Das Anwachsen der Grenzschicht in und hinter einer Expansionswelle. Ing.-Arch. 25 (1957), 155–163.

The momentum integral equation is applied to the problem of the unsteady boundary layer obtained by a rarefaction wave advancing along a plane wall. Both laminar and turbulent cases are considered. The analysis is simplified under the condition of a low Mach number external flow, and as a result it is assumed that (1) the density within the boundary layer is unchanged from its value in the external flow, (2) the form parameter (displacement thickness/momentum thickness) is constant, and (3) the shear stress at the wall is a function of the boundary layer thickness only. It is shown that a first approximation of the solution obtained near the head of the rarefaction wave is the boundary layer flow (laminar or turbulent) over a flat plate with a constantly accelerated external flow.

P. Chiarulli.

Sanyal, Lakshmi. Three-dimensional boundary layer equations. Z. Angew. Math. Mech. 37 (1957), 169–177. (German, French and Russian summaries)

Following Schmidt and Schröder [Luftfahrtforschung 19 (1942), 65–97; MR 4, 120], the three-dimensional boundary layer equations on a surface have been obtained in a rigorous manner with a minimum number of hypotheses which have been precisely stated. For this purpose, the parallel system of coordinates is used and the equations of the boundary layer deduced in an elegant form in these coordinates. (From the author's summary.)

R. DiPrima (Troy, N.Y.).

Covert, Eugene E. On a boundary condition for the calculation of the internal temperature rise due to aerodynamic heating. Naval Supersonic Laboratory, Massachusetts Institute of Technology, Tech. Rep. 202 (1957), iii+19 pp.

This is a discussion of the relative merits and cases of validity of various approximate methods of determining the changing temperature distribution in a body in an airstream of greater "total temperature", given the distribution along the surface of the aerodynamic heat transfer coefficient and taking into account heat conduction within the body.

M. J. Lighthill.

Morduchow, Morris; Grape, Richard G.; and Shaw, Richard P. Stability of laminar boundary layer near a stagnation point over an impermeable wall and a wall cooled by normal fluid injection. NACA Tech. Note no. 4037 (1957), 56 pp.

Feldman, Saul. On the hydrodynamic stability of two viscous incompressible fluids in parallel uniform shearing motion. J. Fluid Mech. 2 (1957), 343–370.

This paper investigates the hydrodynamic stability of two contiguous viscous incompressible fluids, the fluid on one side of the plane interface being bounded by a solid wall and that on the other side extending to infinity. The fluids are in steady uni-directional motion parallel to the interface with a uniform rate of shear in each fluid. Small disturbance theory leads to a pair of linear equations analogous to the Orr-Sommerfeld equation. The conditions for stability depend on the ratio of the viscosities m, the ratio of the densities r, the Froude number F and the Weber number W, as well of course, as the wave length, wave velocity and Reynolds number for the inner fluid. General agreement between the theoretical and experimental values exists for all the critical quantities except the Reynolds number. In particular, gravity and surface tension are found to have a destabilising effect on the flow in agreement with the experimental amounts.

G. Temple (Oxford).

★**Covert, Eugene E. The stability of binary boundary layers.** Massachusetts Institute of Technology, Naval Supersonic Laboratory, Cambridge, Mass., Tech. Rep. 217, June, 1957. v+30 pp.

A boundary layer consisting of a mixture of two components is considered. The linearized disturbance equations are approximated in the usual manner for boundary-layer flow. From an examination of asymptotic solutions of a special type, it is concluded that the stability characteristics can be calculated in the usual way, with the concentration entering only in the determination of the steady-flow velocity and density profiles. Numerical results (based on the inviscid solutions) are presented for the critical wall temperatures required for complete stabilization with respect to two-dimensional disturbances. With helium injection, the critical temperatures are lower for small Mach numbers and higher for large Mach numbers than they are with no injection. {The reviewer notes that there is a term depending on the concentration omitted from the author's perturbation equation of state (eq. 7e, p. 3). For reasons too lengthy to present here, this omission does not affect the numerical results described above. It might be significant in the determination of other stability characteristics.}

D. W. Dunn.

★**Hartunian, R. A.; and Sears, W. R. On the instability of small gas bubbles moving uniformly in various liquids.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 23–42. Stanford University Press, Stanford, Calif. $7.50.

This is an experimental and theoretical study of the instability of small gas bubbles arising through various liquids. The experiments suggest that there are two separate criteria for predicting onset of instability. A critical Reynolds number of 202 appears to be valid for bubbles moving in liquids containing impurities and in the somewhat more viscous liquids. The critical Weber number of 1.26 seems to be valid for bubbles moving in pure and relatively inviscid liquids. The theoretical model based on the general perturbations of a deformed sphere moving uniformly in an inviscid, incompressible fluid give a qualitative indication that the interaction of surface tension and hydrodynamic pressure may result in instabilities. The large deformations in the shape of

the bubble prior to the onset of unstable motion are examined by approximate methods. *G. Temple.*

Movčan, A. A. On the stability of a panel moving in a gas. Prikl. Mat. Meh. **21** (1957), 231–243. (Russian)

★ **Mager, Artur. Transformation of the compressible turbulent boundary layer.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 85–98. Stanford University Press, Stanford, Calif. $7.50.

The equations of motion for a compressible turbulent boundary layer are transformed into their incompressible equivalent, defining a stream function to include mass flow by fluctuating movements and assuming that the Reynolds shear stress per unit mass is the same at corresponding points in the compressible flow and its incompressible equivalent. The transformation is generally similar to Stewartson's transformation [Proc. Roy. Soc. London. Ser. A. **200** (1949), 84–100; MR **11**, 553] but allows an arbitrary dependence of viscosity on temperature. Using the transformation, established knowledge of incompressible boundary layers is applied to the prediction of skin friction and separation in compressible layers, in fair agreement with available measurements. Finally, the transformation is applied to predict the development of self-preserving boundary layers in a temperature gradient and to Maskell's method of computing boundary layer development. *A. A. Townsend.*

Grant, Alison M. A corrected mixing-length theory of turbulent diffusion. J. Meteorol. **14** (1957), 297–303.

It is pointed out that the ordinary from of the mixing length theory is unsuited to the description of the initial stages of turbulent diffusion, owing to the implicit assumption that the duration of an elementary mixing process is zero. By considering the effect of a finite dispersion time, τ, it is shown that the diffusion coefficient, K, at time t after the start of a diffusion process is

$$K=u^2t(1-t/2\tau) \quad \text{for} \quad t<\tau$$
$$K=u^2t \quad \text{for} \quad t>\tau$$

a result qualitatively similar to that obtained from the statistical theory of turbulent diffusion. If allowance is made for a possible distribution of dispersion times, other variations of K with time are found, all preserving the same general behaviour. Finally, the dispersion of smoke from a source in the earth's boundary layer is discussed, and the effect of a finite dispersion time on the theoretical solutions based on an effective diffusion coefficient is considered. *A. A. Townsend.*

Coles, Donald. Remarks on the equilibrium turbulent boundary layer. J. Aero. Sci. **24** (1957), 495–506.

The starting points of this paper are (*a*) the similarity law for turbulent flow near a solid boundary, $u/u_\tau=f(yu_\tau/\nu)$ (u is the local mean velocity, u_τ is the friction velocity, y distance from the wall, ν kinematic viscosity) and (*b*) the defect law, $u_1-u=u_\tau F(y/\delta)$ (u_1 is free-stream velocity, δ is layer thickness) valid in equilibrium (i.e. self-preserving) layers. It is shown that (*a*) and (*b*) imply that

$$P=\frac{\delta^*}{\delta}\frac{d(\log(\delta^*u_1/\nu))}{d(\log u_1/u_\tau)}$$

is constant in any one layer, and it is suggested, from analogy with the known results for self-preserving laminar layers, that $D=d(\log u_1)/d(\log u_\tau)$ is also constant in turbulent flow. The von Kármán equation for the momentum integral is then applied to calculate the development of a layer. The particular cases of flow without wall friction and flow at constant friction coefficient are discussed in some detail. Experimental material from a variety of sources is quoted in favour of the fundamental assumptions and the conclusions derived from them.

A. A. Townsend (Cambridge, England).

Blahó, M.; and Faragó, T. Determination of higher order partial derivatives of the velocity function from known values along a streamline. Acta Tech. Acad. Sci. Hungar. **16** (1957), 363–370. (German, French, and Russian summaries)

Explicit formulas for the partial derivatives of arbitrary order of the velocity potential of a steady isentropic two-dimensional irrotational flow are given by the method of Wu and Brown [J. Aero. Sci. **19** (1952), 183–196]. *C. Saltzer* (Cleveland, Ohio).

Sagomonyan, A. Ya. Penetration of narrow wedge into a compressible fluid. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 2, 13–18. (Russian)

Consider an infinite wedge of small vertex angle 2β which is translated at constant Mach number M_0. At time $t=0$ let it enter, symmetrically, a half-space of compressible fluid. The author seeks the corresponding "automodel" or progressive wave solution of the equations for linearized unsteady plane potential flow, in which the velocity components are functions of x_0/at and y_0/at, where a is the undisturbed velocity of sound. For either sub- or supersonic M_0 this leads eventually for one velocity component to a Dirichlet problem for Laplace's equation on a quadrant of a unit circle. From the known solution the pressure distribution on the wedge and the retarding force can be determined as functions of M_0 and β. *J. H. Giese* (Aberdeen, Md.).

Bagdoev, A. G. Penetration of pressure to the bottom of a compressible fluid. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **11** (1956), no. 2, 19–26. (Russian)

Let z, r be cylindrical coordinates. Suppose $z<0$ is occupied by stagnant compressible fluid. Suppose further that for given $r^*(t)$ and $p_1(r, t)$ the pressure at the free surface is $p(r, 0, t)=p_1(r, t)$ for $0<r<r^*(t)$, and $p(r, 0, t)=0$ for $r>r^*(t)$. In a linearized treatment this leads to the determination, by Kirchhoff's formula, of a cylindrically symmetric solution $\varphi_0(r, z, t)$ of the wave equation such that $p(r, 0, t)=-\rho\partial\varphi_0(r, 0, t)/\partial t$, where ρ is the undisturbed density. In the corresponding problem for a fluid of finite depth h there is an additional boundary condition $\partial\varphi(r, h, t)/\partial z=0$. The new problem is solved by means of Laplace transforms and yields $\varphi(r, z, t)$ as a sum of terms $\varphi_0(r, z_m, t)$, where the choices of z_m correspond to successive reflections of the pressure wave at $z=h$ and $z=0$. *J. H. Giese* (Aberdeen, Md.).

Gustafson, W. A.; and v. Krzywoblocki, M. Z. On multiplicity theorems and an exact solution in diabatic flow. I. Acta Phys. Austriaca **11** (1957), 131–146.

The Munk-Prim substitution principle [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 137–141] relates any steady gas flow to an unlimited number of others whose streamlines

and pressure fields are identical, but whose densities differ by an arbitrary constant along each streamline, the speeds being correspondingly modified so that dynamic pressures are unchanged. This leads to a canonical form of the equations of gas dynamics. These results are here extended to flows in which energy is being added continuously over the field ("diabatic" flows). A cursory examination is also made of analogous results for flows having similar rather than identical patterns, the radii of curvature differing by a constant along related streamlines. *M. D. Van Dyke* (Los Altos, Calif.).

Ščapov, N. M. On Bernoulli's equation for slightly compressible fluids. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 2, 117–119. (Russian)

Musatov, V. V. Remarks on the article by N. M. Scapov "On Bernoulli's equation for slightly compressible fluids". Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 2, 119. (Russian)

Hughes, W. F.; and Osterle, J. F. On the adiabatic Couette flow of a compressible fluid. Trans. A.S.M.E. **79** (1957), 1313–1316.

An investigation is made of the adiabatic flow of air between two parallel plates very close together in relative tangential motion with a pressure differential across the moving plate. *From the authors' summary.*

Jordan, P. F. Aerodynamic flutter coefficients for subsonic, sonic and supersonic flow (linear two-dimensional theory). Aero. Res. Council, Rep. and Memo. no. 2932 (1953), 54 pp. (1957).

Omara, Mohamed Ali. On the subsonic flow of a compressible fluid past fixed cylinders. Proc. Math. Phys. Soc. Egypt **5** (1955), no. 3, 1–21 (1957).

The Imai and Aihara formula [Aero. Res. Inst., Tokyo Imp. Univ., Rep. no. **194** (1942)] for the higher order approximations of the Rayleigh-Janzen M^2-expansion for the velocity potential of a steady, irrotational, two-dimensional, isentropic flow of a uniform stream over an arbitrary cylinder is applied to various problems. A procedure is developed which in some ways simplifies the lengthy analytical computations necessary to obtain these higher approximations. Formulae (most of which have previously been found) are developed for the circular cylinder, elliptic cylinder, and arbitrary profiles with sharp trailing edges. *P. Chiarulli* (Chicago, Ill.).

Yoshihara, Hideo. On the flow over a finite wedge in the lower transonic region. J. Aero. Sci. **24** (1957), 661–666.

The flow over a two-dimensional finite wedge at zero angle of attack for subsonic free stream Mach number is computed using the von Kármán-Guderley transonic approximation. The location of the terminating shock wave as well as the pressure distribution is computed.

Author's summary.

Grodzovskiĭ, G. L. Some special properties of flow around bodies with high supersonic velocities. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 86–92. (Russian)

This paper is concerned with the utility of the approximate rule that pressure on a body in hypersonic flow is proportional to $\sin^2\omega$, where ω is the local angle of deflection of the flow. The constant of proportionality is left free, so that all pressures are referred to the value at one point, such as the tip, where it is presumed known. The rule is closely verified in comparison with numerical solutions for cones and airfoils, and tests of axisymmetric ogives. Its simplicity leads to closed expressions for the drag of various bodies, and to the determination of plane and axisymmetric shapes having minimum drag under various auxiliary conditions. As a final test, the solution for a body of radius proportional to $x^{3/4}$ in a stream of infinite Mach number with $\gamma=1.4$ is carried out by analogy with the unsteady motion produced by a cylindrical piston. The latter problem possesses a self-similar solution that is computed numerically. The drag is found to be 27 per cent less than that of a cone of the same length, whereas the $\sin^2\omega$ rule predicts only a 16 per cent reduction. *M. D. Van Dyke.*

Černyĭ, G. G. Flow around bodies of an ideal gas with great supersonic velocity. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 6, 77–85. (Russian)

A body exposed to a hypersonic stream is surrounded by a thin envelope of dense gas. Plane or axisymmetric shapes are here solved in a "shock-layer" approximation by expanding all flow quantities in powers of $\varepsilon=(\gamma-1)\cdot(\gamma+1)^{-1}$, where γ is the adiabatic exponent. (It is not made clear that the flow is actually analytic in ε only if the bow shock wave is attached). Using the stream function and distance along the surface as independent variables, general expressions are derived for the first two terms. Detailed results are given for an open-nosed cone frustum. The approximations for the wedge and cone are deduced therefrom (and the third term exhibited for the drag at infinite Mach number), and compared with the full inviscid solutions. *M. D. Van Dyke* (Los Altos, Calif.).

Kadyrov, S. Motion of a wing of finite span with supersonic speed. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 8, 35–40. (Russian)

This describes a method for calculating power series solutions for non-linearized steady flows about prescribed bodies with attached shocks. *J. H. Giese.*

Legendre, Robert. Ecoulement supersonique autour d'une aile Δ à bords d'attaque subsoniques. C. R. Acad. Sci. Paris **245** (1957), 889–891.

Le potentiel de la vitesse de perturbation est déduit, par des quadratures, des solutions de problèmes de Dirichlet. *Résumé de l'auteur.*

Kogan, M. N. On bodies of minimum resistance in supersonic gas flow. Prikl. Mat. Meh. **21** (1957), 207–212. (Russian)

Hurrell, Herbert G. Analysis of shock motion in ducts during disturbances in downstream pressure. NACA Tech. Note no. **4090** (1957), 11 pp.

The effect of small downstream pressure disturbances on the position of a normal shock in a duct with area variation is analyzed. For the analysis, the gas flow is treated as quasi-one-dimensional, and boundary layer is neglected. *From the author's summary.*

★ **Greber, Isaac; Hakkinen, Raimo J.; and Trilling, Leon. Some problems of laminar boundary layer shock wave interaction.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 138–158. Stanford University Press, Stanford, Calif. $7.50.

This paper reports the results of some experimental and theoretical studies of the interaction of oblique shock waves with laminar boundary layers.

From the authors' summary.

★ **Bradley, Joe C. General properties of normal shock waves at hypersonic speeds.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 211–230. Stanford University Press, Stanford, Calif. $7.50.

Bhatnagar, P. L.; and Lakshmana Rao, S. K. Problems on the motion of non-Newtonian viscous liquids. Proc. Indian Acad. Sci. Sect. A. **45** (1957), 161–171.

The authors consider incompressible viscous fluids defined by the equation $t_j^i=-p\delta_j^i+F_1d_j^i+F_2d_\alpha^id_j^\alpha$, where t_j^i is the stress tensor, d_j^i is the rate of deformation tensor, and F_1 and F_2 are coefficients which they tacitly take as constants. For plane motion, $(d_\alpha^id_j^\alpha)_{,i}$ is a gradient, and hence the mechanism of generation and transfer of vorticity is independent of the cross-viscosity F_2 and is the same as in an ordinary viscous liquid. Hence a number of the semi-inverse methods used for finding plane solutions of the Navier-Stokes equations will apply also when F_2=const=0. The authors observe also that the power dissipated by cross-viscosity, namely $F_2d_\alpha^id_j^\alpha d_i^j$, is not of one sign, but they appear to be unaware of the definitive work of Baker and Ericksen on this and related problems [J. Washington Acad. Sci. **44** (1954), 33–35; MR **16**, 89]. *C. Truesdell* (Bloomington, Ind.).

★ **Steketee, J. A. An introduction to the equations of magnetogasdynamics.** Institute of Aerophysics, University of Toronto, Review no. 9, April, 1957. vi+35 pp.

A loosely formulated heuristic presentation of the non-relativistic equations of fluid magnetics; for aerodynamicists. *A. A. Blank* (New York, N.Y.).

Chandrasekhar, S.; and Kendall, P. C. On force-free magnetic fields. Astrophys. J. **126** (1957), 457–460.

A method of deriving the general solution of the equation curl $\mathbf{H}=\alpha\mathbf{H}$, where α is constant, is given, the solution being found as the sum of a poloidal and a toroidal vector field. Applying this solution to force-free magnetic fields between spherical boundaries, we find that, as in the axisymmetric case, there is, quite generally, equipartition of energy between poloidal and toroidal components of the field. Finally, it is pointed out that any force-free field of this type can decay freely in a fluid without causing any motion of the fluid. (Authors' summary). *K. C. Westfold* (Pasadena, Calif.).

Beran, Mark Jay. Dispersion of soluble matter in flow through granular media. J. Chem. Phys. **27** (1957), 270–274.

This paper treats the problem of dispersion of soluble matter within a one dimensional tube filled with granular media. It is supposed that the solvent is flowing slowly through the tube with mean velocity $\bar{v}$, and it is desired to find the future concentration pattern of the soluble matter. It is found that after a long time an initially thin slug of soluble matter is normally distributed about the position $\bar{v}t$ and that its standard deviation is proportional to $t^{\frac{1}{2}}$. It is further shown when this effect need not be considered in Ion Exchange problems.

Author's summary.

See also: Functions of Complex Variables: Fil'čakov. **Differential Geometry:** Lagally. **Partial Differential Equations:** Lavrent'ev. **Statistical Thermodynamics and Mechanics:** Müller. **Elasticity, Plasticity:** Mann-Nachbar.

Optics, Electromagnetic Theory, Circuits

★ **Brouwer, Willem. The use of matrix algebra in geometrical optics.** Dissertation, Technical University, Delft, 1957. ii+77 pp.

The author discusses in his dissertation certain problems treated by matrix methods. Among them are the problems of ray tracing through plane and sphere, paraxial ray tracing; formulae for investigating the effect of decentering a lens, and the theory of third-order aberrations.

The author bases his work on the investigation of T. Smith, who, in his work, has made powerful use of matrix methods for optical problems. There are, however, many more optical problems which have successfully been attacked and simplified by matrix methods, for instance, the image of the neighborhood of a general ray in an optical system up to the third order, or the investigation of the problems of color correction.

M. Herzberger (Rochester, N.Y.).

Smith, T. The optical imagery of curved surfaces. Proc. Roy. Soc. London. Ser. A. **240** (1957), 458–461.

The general form of the angle characteristic of Hamilton is obtained for aberration-free imaging of a curved rotationally symmetric surface onto another such surface. A succession of imageries of this type is also considered and formulae are given relating to the expansion coefficients of the characteristic function of the combination in terms of the expansion coefficients relating to the individual surfaces. The results have also a bearing on computations relating to systems which are not free of aberrations. *E. Wolf* (Manchester).

Kontorovitch, M. I.; and Murav'ev, U. K. Derivation of the laws of reflection of geometrical optics on the basis of an asymptotic treatment of the diffraction problem. Translated by A. Shenitzer, with a foreword by M. Kline. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. EM-103 (1957), ii+23 pp. Translated from Ž. Tehn. Fiz. **22** (1952), 394–407.

Arakengy, Albert. Liouville's theorem and the intensity of beams. Amer. J. Phys. **25** (1957), 519–525.

The meaning of Liouville's theorem on the flow of points in phase space is discussed. The intensity for particles or for radiation is defined in the most general case. It is shown that the familiar theorem which states that an optical instrument cannot change the brightness of any object which it images is a special case of Liouville's theorem. The generalization of this result to classical particle- and radiation-handling systems of many kinds is discussed. The result holds even for situations in which the fields or refractive index distributions are arbitrarily varying in time, though in such situations the trajectories of "constant brightness" may move around by bending and distorting so that a stationary observer may thereby see some changes. The problem of point sources is considered. Simple examples of phase space motions are exhibited. *Author's summary.*

Tetelbaum, S. I.; and Zmorovich, V. A. On the possibility of improving the clearness of images produced by optical instruments. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 323–327. (Ukrainian. Russian and English summaries)

A method is suggested for obtaining a clearer image of an object through its less distinct image produced by optical instruments. *Authors' summary.*

Buckingham, A. D.; and Stephen, M. J. A theory of the depolarization of light scattered by a dense medium. Trans. Faraday Soc. **53** (1957), 884–893.

A general theory of the depolarization of the light scattered elastically by an assembly of interacting molecules is developed. For spherical molecules with dimensions small relative to the wavelength of the light, and for frequencies well below those of the electronic absorption region, a relationship, valid at all densities, between the depolarization and the molecular refraction is established. *From the authors' summary.*

Paschke, F. On the nonlinear behavior of electron-beam devices. RCA Rev. **18** (1957), 221–242.

The nonlinear space-charge-wave equation is derived. By third-order successive approximation this complicated equation is split into three simultaneous linear differential equations, which are solved for the case of a velocity-modulated electron beam (klystron). When excited at the fundamental frequency ω, the beam produces all harmonics with the frequency $n\omega$. At fairly low levels the maximum current amplitudes are found to be proportional to the $(1-n/2)$th power of the direct current. Equations for the gain and efficiency of a two-cavity klystron are given. The saturation efficiency is computed to be about 55 per cent. Thus the space-charge, or "longitudinal debunching," does not reduce the efficiency of a klystron appreciably below the well-known ballistic theoretic value of 58.2 per cent. Furthermore, it is shown that a computation of kinetic power flow, using the linear solutions, is not quite justifiable. With the simultaneous linear differential equations given, it is possible to treat nonlinear phenomena in traveling wavetubes. *Author's summary.*

Zin, Giovanni. Teoria analitica del campo elettromagnetico. Ann. Mat. Pura Appl. (4) **43** (1957), 215–259.

In dieser Arbeit wird die Berechnung elektromagnetischer Felder auf die Lösung einer Differentialgleichung erster Ordnung für einen Vektor $K=E+\lambda H$ zurückgeführt, wo E und H bezw. die Vektoren der elektrischen und der magnetischen Feldstärke bezeichnen, während λ eine skalare Grösse ist, welche die Frequenz, die Leitfähigkeit, die Permeabilität und die Dielektrizitätskonstante enthält. Die Differentialgleichung für K lautet: rot $K+mK=\phi$, wo m eine Skalare ist, die λ, die Leitfähigkeit, die Frequenz und die Dielektrizitätskonstante enthält, während ϕ einen Vektor darstellt, dessen Richtung mit derjenigen der elektrischen Konvektionsstromdichte j zusammenfällt: $\phi=\lambda j$. Zur Lösung der Differentialgleichung deutet der Verfasser zwei Wege an, einen Integralweg und einen zweiten, der auf der Entwicklung von K in eine Reihe geeigneter Funktionen beruht. In dieser Arbeit wird nur der erste Weg beschritten. Es wird ein Integralausdruck für K abgeleitet, aus dem der Wert von K in einem regulären Punkt eines Gebietes hervorgeht, wenn die K-Werte auf der Berandung dieses Gebietes bekannt sind. Eine notwendige und hinreichende Bedingung wird abgeleitet, dafür dass eine stetige Vektor-funktion K in einem Gebiet V mit dem Rande S existiert, welche überall in $V-S$ der Gleichung rot $K+mK=0$ genügt und die auf dem Rande S die vorgegebenen Werte annimmt. Für die Probleme des elektromagnetischen Feldes werden Integralgleichungen abgeleitet. Als Sonderfälle werden die statischen Felder behandelt, welche auf das Gleichungssystem rot $K=\phi$, div $K=f$ (f eine Skalare) führen.

M. J. O. Strutt (Zürich).

Zăgănescu, M. Über die hereditäre Elektrodynamik. Z. Physik **148** (1957), 508–512.

The integro-differential equations of hereditary electrodynamics established by V. Volterra [Leçons sur les fonctions de lignes, Gauthier-Villars, Paris, 1913, p. 118] for the treatment of electrical and magnetic after effects are written in tensor notation. The resulting equations contain integrals over a time variable; these are not covariant when submitted to a Lorentz transform because the time element is not Lorentz invariant. But covariance can be achieved by a generalization of these equations in which the time element is replaced by an element of Minkowski space that is Lorentz invariant. Integration now extends over the Minkowski space. *B. Gross.*

★ **Flammer, Carson. Variational formulae for domain functionals in electromagnetic theory.** Air Force Cambridge Research Center, Tech. Rep. 57. Stanford Research Institute, Menlo Park, Calif., Jan., 1957. iii+23 pp.

Variational formulas of the Hadamard type are derived for solutions of the vector Helmholtz equation of electromagnetic theory. Formulas are found for the first and second variations of the scattering cross section, of the dyadic Green's function, and of field components due to electric and magnetic current sources. It is explained how the method can be extended to yield higher order variations of the scattering cross section. *P. R. Garabedian.*

Wilcox, C. H. An expansion theorem for electromagnetic fields. Comm. Pure Appl. Math. **9** (1956), 115–134.

An integral representation is given for fields $\mathbf{A}(\mathbf{r})$ that satisfy the vector Helmholtz equation and the Silver-Müller radiation condition in an exterior domain. From this an expansion theorem for $\mathbf{A}(\mathbf{r})$ is derived (r,θ,ϕ are spherical coordinates)

$$\mathbf{A}(\mathbf{r})=\frac{e^{ikr}}{r}\sum_{n=0}^{\infty} r^{-n}\mathbf{A}_n(\theta,\phi)$$

valid for $r>c$ and absolutely and uniformly convergent in r,θ,ϕ in $r\geqq c+\varepsilon$.

The vector coefficients $\mathbf{A}_n(\theta,\phi)$ ($n\geqq 1$) in this expansion are determined by $\mathbf{A}_0(\theta,\phi)$ (the radiation pattern) through recurrence relations. The relations of vector to scalar radiation function are discussed. *F. Oberhettinger.*

Skalskaya, I. P. The electromagnetic field of a dipole located in the interior of a parabolic reflector. Translated by A. Shenitzer, with an appendix by W. Magnus and N. N. Lebedev. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. EM-103 (1957), ii+22 pp.

Translated from Ž. Tehn. Fiz. **25** (1955).

Kischel, Kurt. Zur Theorie elektrischer Wellen in inhomogenen Plasmen. Ann. Physik (6) **19** (1956), 309–321 (1957).

Power, G. Simple expressions for the two-dimensional Maxwell stress distribution in ideal dielectrics. Amer. J. Phys. **25** (1957), 344–346.

By use of the complex variable, an attempt is made to obtain simple expressions for the Maxwell stress components in the case of two-dimensional electrostatic fields, when the dielectric material is homogeneous and effects of electrostriction can be neglected. Examples are given

to illustrate how the results may be interpreted, with emphasis on the stress distribution at the interface separating two ideal dielectrics. *Author's summary.*

Aharoni, A.; Frei, E. H.; Shtrikman, S.; and Treves, D. The reversible susceptibility tensor of the Stoner-Wohlfarth model. Bull. Res. Council. Israel Sect. A. **6** (1957), 215–238.

★ **Wait, James R. The propagation of V. L. F. pulses to great distances.** National Bureau of Standards, Rep. 5513, Boulder, Colo,, Sept., 1957. 56 pp.

A theoretical study is presented for the propagation of electromagnetic pulses at very low frequencies to large distances. The space between the earth and the ionosphere is represented as a wave guide with sharply bounded and concentric spherical boundaries. The concept of phase and group velocity and its application to the present problem is discussed in some detail. The influence of the propagation medium on the shape of the envelope of a quasi-monochromatic pulse is also considered. Using an alternative approach, the response of an impulsive source is also calculated and is shown to be a damped oscillatory function of time with a quasi half-period varying in a predictable manner with distance of travel in agreement with the observations of Norinder and Hepburn. *Author's summary.*

Haselgrove, J. Oblique ray paths in the ionosphere. Proc. Phys. Soc. Sect. B. **70** (1957), 653–662.

An electronic digital computer has been used to calculate ray paths in a parabolic ionospheric layer when the waves are incident in the plane of the magnetic meridian. This is a special application of a very general method. The results of the calculations are compared with what would be deduced from vertical soundings by the standard approximations of using Martyn's and Breit and Tuve's theorems as if there were no field.

Author's summary.

Yamada, Ryozo. On the radio wave propagation in a stratified atmosphere. II. J. Phys. Soc. Japan **12** (1957), 1022–1030.

Van-Loc, Phan. Diffraction des ondes lumineuses par une fente indéfinie à bords parallèles et par un demi-plan. C. R. Acad. Sci. Paris **244** (1957), 1470–1473.

The Maggi-Rubinowicz transformation reduces the Kirchhoff diffraction integral to a line integral along the edge of the diffracting body. Strictly, the transformation is only applicable when the edge is closed, but several cases of interest which do not obey this condition (e.g. diffraction by a half-plane) may be treated as limiting cases. In this note several cases of diffraction from point sources and line sources are considered in this way. *E. Wolf* (Manchester).

Gosar, P. A new method for solving multiple scattering problems in inhomogeneous media. Nuovo Cimento (10) **5** (1957), 1437–1455.

This paper is the continuation of a previous work [Nuovo Cimento (10), **4** (1956), 688–702; MR **19**, 211], where the author calculated the propagation of scalar waves through a medium in which domains of slightly different dielectric constant act as scattering centers. Here the same problem is treated with the following assumptions: (1) no absorption, and variations in the dielectric constant are small; (2) the scattering domains are equal and spherically symmetric and large compared to a wave length; (3) they are distributed at random with low density. The assumption in the previous paper, that the average refractive index in each domain should be equal to the total average over the medium, is now dropped. Again the basic idea is to describe the wave field as a "macrostructural wave", which takes the overall change in refractive index into account, plus scattered waves. As a special case the multiple scattering by spheres of constant dielectric constant is treated.

N. G. van Kampen (New York, N.Y.).

Millar, R. F. The diffraction of an electromagnetic wave by a large aperture. Proc. Inst. Elec. Engrs. C. **104** (1957), 240–250.

With the aid of methods previously developed [same Proc. **103** (1956), 177–185; **104** (1957), 87–95; MR **17**, 920] an approximate expression is obtained for the difference from unity of the transmission coefficient for plane waves at normal incidence of a large aperture of rather general form in a plane perfectly-conducting screen. The coefficient is expressed in the form of one or more line integrals taken around all (or part) of the aperture rim. These may be evaluated analytically in certain cases, e.g. a circular or elliptical aperture. In the former case the result agrees with that derived elsewhere [see the second paper cited above]; the ellipse is considered in some detail in the present paper, and the transmission coefficient is evaluated for a range of eccentricities. The results appear to be in qualitative agreement with the limited experimental data available. *Author's summary.*

Phariseau, P. The diffraction of light by two perpendicular ultrasonic waves. Physica **23** (1957), 651–656.

We establish a system of difference-differential equations describing the diffraction of light by two mutually perpendicular supersonic waves. When the two supersonics have intensities of the same order of magnitude, we prove that the integration of these equations is reduced to the integration of two systems which are of the Raman-Nath type for the diffraction by one ultrasonic wave. If the two ultrasonics have very different intensities, the equation becomes rather complicated. In this case it is still possible to reduce their integration to that of a system of the Raman-Nath type and to a more complicated one. Approximate solutions will be established in a following paper. *Author's summary.*

Chu, Chiao-Min; Clark, George C.; and Churchill, Stuart W. A six flux solution of multiple scattering by dispersion with a partially reflecting boundary. J. Phys. Chem. **61** (1957), 1303–1309.

The scattering of electromagnetic radiation by dense dispersions having a partially reflecting boundary was investigated theoretically. Finite, parallel-plane dispersions of non-absorbing, uniformly sized particles were considered. A six-flux representation was used to reduce the transport equation to a set of ordinary differential equations. An analytical solution was found for these equations and the appropriate boundary conditions for reflection. The intensity in any direction and the power intercepted by any arbitrarily oriented plane at any location within the dispersion can be obtained by resolution of the discrete components. The solution indicates a bilinear effect for reflection upon the discrete components of the intensity. The effect of the other variables is illustrated graphically. *Authors' summary.*

Schouten, J. P.; et de Hoop, A. T. Sur la réflexion d'une onde électromagnétique plane par une surface rugueuse parfaitement conductrice. Ann. Télécommun. **12** (1957), 77–80.

Jauquet, Christian. Excitation d'une onde de surface transverse magnétique se propageant sur un cylindre diélectrique. Ann. Télécommun. **12** (1957), 217–233.

Weinberg, Louis. Explicit formulas for Tschebyscheff and Butterworth ladder networks. J. Appl. Phys. **28** (1957), 1155–1160.

Green found the closed-form formulas for the element values in a resistance-terminated ladder network that has a maximally flat (Butterworth) or equal-ripple (Tschebyscheff) characteristic. The Green formulas apply only when all the zeros of the reflection coefficient ρ are chosen to lie in one half-plane. In this paper we present new formulas for the element values for the case in which the zeros of ρ are chosen to alternate in the left and right half-planes. These formulas apply for n odd, where n is the degree of the denominator of the transfer function, and for any nonzero ratio of the output to the input resistance. The networks obtained are related to the symmetrical ones given by Bennett's and Belevitch's formulas for matched networks. *Author's summary.*

Bolinder, E. Folke. Impedance transformations by extension of the isometric circle method to the three-dimensional hyperbolic space. J. Math. Phys. **36** (1957), 49–61.

This paper shows that for loxodromic transformations, corresponding to impedance transformations through lossy networks, the natural space to which the isometric circle method should be transferred is the three-dimensional hyperbolic space which has the Riemann unit sphere as its absolute surface. Two important problems in network theory are studied in this space: (1) analysis or synthesis of a lossy two terminal-pair network from three measurements, and (2) cascading of lossy two terminal-pair networks. *From the introduction.*

Cruickshank, A. J. O. Kron's solution of orthogonal networks. Matrix and Tensor Quart. **7** (1956), 51–55.

The method of Kron for the analysis of linear lumped constant networks [Tensor analysis of networks, Wiley, New York, 1939] is compared to the classical method of circuit analysis. *C. Saltzer* (Cleveland, Ohio).

★**Angot, André. Compléments de mathématique à l'usage des ingénieurs de l'électrotechnique et des télécommunications. Préface de Louis de Broglie.** 3me éd. Editions de la Revue d'Optique, Paris, 1957. viii+836 pp. 4800 francs.

Fonctions d'une variable complexe, séries et intégrales de Fourier, calcul vectoriel et tensoriel, algèbre matricielle, équations différentielles et aux dérivées partielles en coordonnées rectilignes ou curvilignes, étude de fonctions importantes, notamment celles de Bessel et de Legendre, analyse symbolique, ... principes du calcul des probabilités ... aperçu sur les fonctions aléatoires. ... Et toujours les applications à des problèmes de circuits électriques.

Cet ouvrage est spécialement écrite pour les ingenieurs electriciens, mais on doit aussi recommander sa lecture aux physiciens. *De la préface.*

Gold, Louis. Space-charge in the relativistic magnetron. J. Electronics Control **3** (1957), 87–96.

Buneman, O. A small amplitude theory for magnetrons. J. Electronics Control **3** (1957), 1–50.

See also: Differential Geometry: Lagally. **Manifolds, Connections:** Nakae (first paper). **Numerical Methods:** Huhrikov. **Structure of Matter:** Yoshioka. **Fluid Mechanics, Acoustics:** Steketee; Chandrasekhar and Kendall. **Relativity:** Arzeliès. **Astronomy:** Poincelot; Eckart.

Classical Thermodynamics, Heat Transfer

Yusupov, K. Yu. Solution of a problem of heat conduction. Kazan. Aviac. Inst. Trudy **29** (1955), 39–46. (Russian)

A problem in the heat transfer arising from electrical resistance heating of a wire is considered. The problem is stated in terms of the heat conduction equation with a distributed heat source, and series solutions satisfying the conduction boundary condition are developed.

N. A. Hall (New Haven, Conn.).

Eckert, E. R. G.; Hartnett, J. P.; and Birkebak, Roland. Simplified equations for calculating local and total heat flux to nonisothermal surfaces. J. Aero. Sci. **24** (1957), 549–551.

The note gives a simplified method for calculating heat transfer across boundary layers to surfaces whose temperatures are non-uniform. Laminar and turbulent boundary-layer flows, with or without pressure gradients, are considered. The note continues and simplifies the work contained in M. Tribus and J. Klein, Heat Transfer Symposium, 1952, pp. 211–235 [Univ. of Michigan Press, 1953].

J. Kestin (Providence, R.I.).

Chang, Yan Po. A theoretical analysis of heat transfer in convection and in boiling. Trans. A.S.M.E. **79** (1957), 1501–1509, discussion 1509–1513.

See also: Numerical Methods: Elrod. **Elasticity, Plasticity:** Nowacki. **Fluid Mechanics, Acoustics:** Philippova; Baron; Covert.

Quantum Mechanics

★**Kramers, H. A. The foundations of quantum theory.** North-Holland Publishing Company, Amsterdam; Interscience Publishers Inc., New York, 1957. xv+228 pp. $6.50.

This contains the first five chapters in translation of Kramers' famous text book [Akademische Verlagsgesellschaft, Leipzig, 1933] and is presumably intended as an introduction to Quantum Theory. The translator has done his work well and has added occasional notes in explanation or to give more modern references. The text is very readable. It is not the simplest possible exposition but could be used at (British) undergraduate level by honours students with a reasonable background of methods of mathematical physics, for whom it should be stimulating reading. After chapters on introductory fundamentals there follow a long chapter on Transformation Theory (hydrogen atom being treated at the end) and one on Perturbation Theory. For many students the

book might serve best as companion to one of the many books which give many applications of the theory while leaving the basic ideas rather ill-defined. *C. Strachan.*

Hugenholtz, N. M. Perturbation theory of large quantum systems. Physica **23** (1957), 481–532.

A perturbation theory is developed for many-particle quantum systems. It is based on the systematic use of the resolvent operator $(H-z)^{-1}$ ($H=H_0+V$=total Hamiltonian, z complex energy parameter) and of its diagonal part in the representation of the unperturbed stationary states. A number of general results and formulae of the resolvent method had been derived previously by the reviewer [Physica **21** (1955), 901–923; **22** (1956), 343–354]. They are here applied and greatly expanded for the particular case of a Fermi gas of low density, whereby the effect of the interparticle interaction (which is treated as perturbation) is studied for the ground state and those excited states for which a finite number of particles are excited. Use is made of diagrams, and the concept of connectedness of diagrams turns out to be a very powerful tool, leading to a complete determination of a number of properties in the limit of a large gas of given density. These properties are the perturbed ground state energy and wave function, and certain energy and wave function properties of the excited states mentioned above, which are not stationary states in presence of the interaction, but manifest a dissipative behaviour. Although applied here to a special system the method has a very general scope. *L. Van Hove* (Utrecht).

Hugenholtz, N. M. Perturbation approach to the Fermi gas model of heavy nuclei. Physica **23** (1957), 533–545.

The general perturbation theory of the Fermi gas presented in the previous paper is here applied to the limiting cases of low and high densities, and its lack of convergence for high densities is established. The treatment recently proposed by Brueckner and Levison for the structure of nuclear matter [see, e.g., Phys. Rev. (2) **97** (1955), 1344–1352] is critically examined and is recognized to be an improved low density approximation, the improvement affecting the single-particle propagation through the medium. One-particle and one-hole excitations of the Fermi gas are shown to be long-lived metastable states when the momentum is near the Fermi momentum. The formula for the optical potential is derived from the general theory. *L. Van Hove.*

★ **Koopman, B. O. Quantum theory and the foundations of probability.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. **VII**, pp. 97–102. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

The author gives a probabilistic analysis of various experiments in quantum mechanics, within the framework of the classical theory of relative probabilities (Boolean algebra). He interprets this as showing that quantum theory calls for no new algebra of logic or probability. *G. Birkhoff* (Cambridge, Mass.).

Novožilov, Yu. V. The variational principle, scale transformation and virial theorem in the relativistic quantum theory. Vestnik Leningrad. Univ. **12** (1957), no. 4, 5–24. (Russian)

A virial theorem is derived for the phase of the Dirac wave-function in the case of a general central force and also in the case of a Coulomb force. These involve not only the derivative of the phase with respect to the energy, but also that with respect to the mass; however, the latter can be neglected in the case of sufficiently high energy. A variational principle is obtained for the scattering amplitude which is a generalization of that obtained by W. Kohn [Phys. Rev. (2) **74** (1948), 1763–1772]. A virial theorem is also derived for the case of a time-dependent field. The invariance of the quantized field equations under change of scale is discussed, and a relation is obtained which can be regarded as a virial theorem. Finally, a variational principle is obtained for the Bethe-Salpeter equation. *N. Rosen* (Haifa).

Potier, Robert. Sur la double covariance (quantique et relativiste) dans la seconde quantification. J. Phys. Radium (8) **18** (1957), 422–433.

Verfasser schließt an die Untersuchungen von Costa de Beauregard an, der in einer Reihe von Arbeiten [C. R. Acad. Sci. Paris **239** (1954), 1357–1359; J. Phys. Radium (8) **16** (1955), 770–780; **17** (1956), 872–875; MR **16**, 431; **17**, 437; **19**, 225] im Rahmen der Wellenmechanik ohne Superquantelung den quantenmechanischen Formalismus explizit in relativistisch kovariante Form gebracht hatte. Die Ergebnisse Beauregards sollen nunmehr in die zweite Quantelung einbezogen und damit für die Quantentheorie der Wellenfelder nutzbar gemacht werden. — Wie bereits in C. R. Acad. Sci. Paris **242** (1956), 470–473, 878–881, 1694–1697, 1961–1964 [MR **17**, 1031, 929] sieht Verfasser einen einfachen Weg zu diesem Ziel in der Einführung eines neuen Kovarianzbegriffes, einer von ihm Quantenkovarianz genannten Forderung, deren Auswirkungen sich denen der relativistischen Kovarianz überlagern und die Quantenkovarianz als eine Verallgemeinerung des Relativitätsprinzips erscheinen lassen. — In der klassischen Mechanik ist eine Partikel in Minkowski's Raumzeit durch einen Momentanpunkt gegeben, welcher der Endpunkt eines Vierervektors ist. Die die Geometrie des Minkowskiraumes bestimmende Gruppe ist die Lorentzgruppe, welcher diese Vierervektoren unterworfen sind. Eine Bewegung der Partikel wird durch ein einparametriges Vektorfeld $X(\tau)$ dargestellt. Vom klassischen kinematischen Aspekt des Problems einer materiellen Partikel gelangt man zu seinem dynamischen Aspekt durch die Annahme, daß das Vektorfeld $X(\tau)$ einer gewissen Differentialgleichung genügt, die gegenüber Lorentztransformationen unempfindlich ist. Das ist das relativistische Kovarianzprinzip. — In der Wellenmechanik erscheint einer Partikel P die ein- oder mehrkomponentige de Broglie'sche Wellenfunktion Ψ (der Raumzeitkoordinanten der Minkowskischen Welt) zugeordnet. Ψ selbst gehört nicht mehr zur Minkowskiwelt sondern zum Hilbertraum (H) der Lösungen der Wellengleichung. Ist $\psi_1, \psi_2, \cdots$ eine unendliche Basis in (H) und

$$\psi_i' = A_i{}^k \psi_k$$

eine zulässige Basistransformation, so ist die in (H) definierte Geometrie durch die Gruppe G aller linearen Transformationen in (H) gegeben, welche die Skalarprodukte und Normen der Funktionen in (H) ungeändert lassen. — Unter der Einwirkung äußerer Ursachen bestehen für P Übergangsmöglichkeiten, die jetzt durch ein einparametriges Vektorfeld $\Psi(\tau)$ im Hilbertraum dargestellt werden. Der kinematische Aspekt des Problems einer Partikel P ist also in der Quantenmechanik dem

der klassischen Mechanik sehr ähnlich, nur daß die Vektoren $\Psi(\tau)$ zu (H) gehören und nicht zur Minkowskischen Raumzeit. Um auch hier wieder zum dynamischen Aspekt zu gelangen, muß man das Vektorfeld $\Psi(\tau)$ in (H) einer "Entwicklungsgleichung" (Entwicklung nach dem Parameter τ) unterwerfen, welche gegenüber Basistransformationen in (H) unempfindlich sein muß. Darin liegt das Quantenkovarianzprinzip.

Sind $\Psi^1, \Psi^2, \cdots$ kontravariante Komponenten des Hilbertvektors Ψ

$$\psi=\Psi^1\psi_1+\Psi^2\psi_2+\cdots,$$

so besteht relativistische Invarianz, was immer die tensorielle oder spinorielle Natur der Funktion ψ sein mag. Desgleichen erweisen sich die Matrizenelemente A_i^k als relativistische Invarianten und damit erscheint Quanten- und Relativitätskovarianz verträglich. Andererseits hatte Verfasser und zugleich auch Costa de Beauregard bereits früher gezeigt, daß man die Metrik in (H) durch passende Wahl des Skalarproduktes zweier Wellenfunktionen relativistisch invariant gestalten kann. Die Gruppe G bleibt unempfindlich gegenüber Lorentztransformationen in Minkowski's Raumzeit. — Nennt man den Wechsel einer Basis in (H) eine Darstellungstransformation, so kommt die Einführung des Quantenkovarianzprinzips in die Theorie darauf hinaus, der "Entwicklungsgleichung" Abhängigkeit von der Wahl der Darstellung zu verbieten. — Unter diesen Voraussetzungen und mit solchen Begriffsbildungen entwickelt Verfasser nunmehr die weitere Quantentheorie der Wellenfelder in folgenden Abschnitten: (2) Vektoren im Hilbertraum, (3) normierte und nichtnormierte Vektoren, (4) Tensoren in (H), (5) Konvergenzprobleme, (6) Partikelsysteme gleicher Art; symmetrische und schiefsymmetrische Tensoren, (7) Wellenfunktionen und Superquantelung, (8) Partikel und Antipartikel, Vertauschungsrelationen, (9) Entwicklungsgleichungen eines Systems, (10) Spin und Statistik, (11) zugeordnete Operatoren, Schrödingerbild. *M. Pinl* (Köln).

Mušicki, Đ. Application du principe de Pfaff en mécanique quantique. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N.S.) **9** (1956), 45–53. (Serbo-Croatian. French summary)

L'auteur démontre que l'équation de Schrödinger pour les états stationnaires peut être déduite du principe de Pfaff tel qu'il a été formulé par Bilimovitch [cf. MR **6**, 243]. À cet effet on part des expressions pour l'énergie cinétique et potentielle des ondes de matière. Dans le cas envisagé, le principe de Pfaff est équivalent au principe du minimum de l'énergie. *Résumé de l'auteur.*

Petiau, Gérard. Sur les fonctions d'ondes associées au mouvement des corpuscules en mécanique ondulatoire. C. R. Acad. Sci. Paris **245** (1957), 293–296.

Les fonctions d'ondes guidées générales sont obtenues à partir d'une solution de l'équation d'ondes combinaison de solutions régulières et singulières représentant la structure du corpuscule dans un repère propre et en utilisant les transformations laissant l'équation d'ondes invariante au cours du temps. *Résumé de l'auteur.*

Barašenkov, V. S. On the impossibility of the Hamiltonian formulation of theory with the form-factor. Nuovo Cimento (10) **5** (1957), 1469–1479.

The validity of the Hamiltonian formulation of nonlocal theories is examined. In the first part of the paper, the author considers a system of oscillators with a nonlocal interaction described by an action function

$$S_{int}=-\tfrac{1}{2}\sum_{ij=1}\gamma_{ij}\int_{-\infty}^{\infty}q_i(t)F_{ij}(t_1-t_2)q_j(t_2)dt_1dt_2.$$

The initial value problem is solved exactly by Fourier transforms, and is shown to be uniquely determined only if the solution satisfies a boundedness condition $\int_{-\infty}^{\infty}|q_i(t)|\exp\{-\beta|t|\}dt\leqq R<\infty$, and then the initial values of q_i, $\dot{q}_i$ are not, in general, sufficient to specify the solution, but higher derivatives may also have to be given initially. If q_i, $\dot{q}_i$ are sufficient, the form factors F_{ij} are said to be "normal". Lagrangian and Hamiltonian functions for the equations of motion are given. (The definitions of L and H in equations (15) and (16) appear incorrect, since the interaction terms proposed are linear in the displacements).

The method of Pauli [Nuovo Cimento (9) **10** (1953), 648–667; MR **15**, 81] is applied to this problem, and the author claims that, for the classical problem, it cannot give anything new as, on the one hand, the complete solution must be known before it can be applied, and on the other hand, no boundedness condition is imposed.

Hayashi's method [Progr. Theoret. Phys. **10** (1953), 533–548; MR **17**, 565] for the Hamiltonian formulation is also examined, and it is claimed that the system of equations so obtained is not integrable in the lowest approximation in powers of the coupling constant. {The reviewer is in agreement with Hayashi. Equation (29) of this paper contains non-charge conserving terms, and does not appear to be correct.} *C. A. Hurst* (Adelaide).

Burton, W. K.; and de Borde, A. H. Derivation of the functional integral formalism for Fermi systems from the canonical formalism. Nuovo Cimento (10) **5** (1957), 1510–1519.

It is well known that the propagation functions of coupled field problems can be written as Feynman history integrals [see, e.g., P. T. Matthews and A. Salam, Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 693; I. M. Gelfand and A. M. Yaglom, Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 77–114; MR **17**, 1261]. When the fields involved satisfy Bose commutation relations, the resulting formal integrals can be made mathematically definite in a variety of simple ways. When the fields involved satisfy anticommutation relations, the history integrals are with respect to anticommuting variables and quite different specifications are natural [see, e.g., I. M. Halatnikov, Ž. Eksper. Teoret. Fiz. **28** (1955), 633–636; MR **17**, 332]. In a previous paper, the authors gave an explicit prescription for the one particle propagator [Nuovo Cimento (10) **4** (1956), 254–269; MR **19**, 219]. However, if applied to many body propagators, the prescription gives wrong results; the present paper proposes an alternative method which works for all propagators. It is closely related to one proposed by Symanzik [Z. Naturf. **9a** (1954), 809–824; MR **16**, 778]. The basic idea is to expand the anticommuting fields in an orthonormal anticommuting set; the functional integrals are then integrals with respect to the (commuting) coefficients of this expansion.

A. S. Wightman (Princeton, N.J.).

Park, David. Diffusion par deux potentiels. C. R. Acad. Sci. Paris **245** (1957), 291–293.

Let ψ satisfy the Schrödinger equation with the Hamiltonian $H_0+V_1+V_2$ and ϕ be its solution when $V_1=$

$V_2=0$; then purely formal manipulation leads to the result that U defined by $\psi=U\phi$ can be written

$$U=U_2(1-t_1t_2)^{-1}U_1=U_1(1-t_2t_1)^{-1}U_2,$$

where $U_1=1+t_1$ and $U_2=1+t_2$ are the two special cases of U in which $V_2=0$ and $V_1=0$, respectively. When expanded in t_1 and t_2, this expression for U corresponds to a certain rearrangement of Feynman diagrams.

F. Rohrlich (Iowa City, Iowa).

Logunov, A. A.; Stepanov, B. M.; and Tavhelidze, A. N. On the role of bound states in processes of photogeneration. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 45–47. (Russian)

The part played by bound states in the dispersion relations of photomeson production processes is investigated. Expressions are obtained which lead to results equivalent to those obtained by G. F. Chew and F. E. Low [Phys. Rev. (2) **101** (1956), 1579–1587]. *N. Rosen.*

Ladányi, K. Zur statistischen Näherung des Mehrteilchenproblems der Wellenmechanik. I. Acta Phys. Acad. Sci. Hungar. **7** (1957), 161–166. (Russian summary)

By using the one-particle wave functions proposed by W. Macke [Phys. Rev. (2) **100** (1955), 992–993], calculations are carried out for the lowest energy of a system consisting of an arbitrary number of non-interacting electrons. The result for the Fermi energy is different from that given by the statistical model of the atom, but goes over into the latter in the case of a large number of particles. *N. Rosen* (Haifa).

Gilvarry, J. J. Asymptotic relations in the Thomas-Fermi-Dirac atom model. J. Chem. Phys. **27** (1957), 150–154.

Asymptotic solutions of the Thomas-Fermi-Dirac equation for the case where the atom radius is vanishingly small (Fermi-Dirac limit) are considered.

From the author's summary.

Bowcock, J. Compound nuclear theory and the optical model. Proc. Phys. Soc. Sect. A. **70** (1957), 515–528.

A formalism is developed to relate the total cross section for low-energy nucleon-nucleus scattering to the cross section due to a potential well. By introducing the intermediate model of Lane, Thomas and Wigner for the compound nucleus states, a simple relation between these two cross sections is obtained. The method of extracting a potential well to replace the hard sphere scattering and the scattering from 'far distant' levels is given and then extended to find the well which describes the average total cross section. *Author's summary.*

Jansen, L. Interactions between permanent multipole moments. Physica **23** (1957), 599–604.

A vector-tensor notation for interactions between arbitrary charge distributions, at large distance from each other, is developed which leads to a simple expression for the expectation value of the Coulomb potential. It is proven that for cylindrically symmetric charge distributions the interactions between permanent moments of any order may be replaced by those between linear distributions of charge along the axes of cylindrical symmetry. *Author's summary.*

Mittlestaedt, P. Über ein optisches Kernmodell für Neutronenreaktionsquerschnitte. Z. Naturf. **12a** (1957), 643–646.

Das optische Modell für Neutronen gestattet die Berechnung des totalen Wirkungsquerschnittes σ_t. Dagegen ist es in dieser Theorie nicht möglich, den Reaktionsquerschnitt σ_r und damit den elastischen Querschnitt σ_e zu bestimmen. Es soll daher eine begrifflich naheliegende Erweiterung des optischen Modells vorgeschlagen werden, die es gestattet, neben σ_t auch σ_r und σ_e in einer dem bisherigen Modell analogen phänomenologischen Weise zu beschreiben. Die Zusammenhänge des ursprünglichen mit dem hier vorgeschlagenen Modell werden auf Grund der physikalischen Interpretation des optischen Modells genauer diskutiert.

Zusammenfassung des Autors.

Greenberg, Howard; and Borowitz, Sidney. A variational calculation of the scattering cross section for nearly zero-energy electrons by hydrogen atoms. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-29** (1957), ii+51 pp.

Sengupta, S. Nuclear electric and magnetic moment in j-j coupling. Indian J. Theoret. Phys. **4** (1956), 33–42.

With the help of the diagonal sum rule, the nuclear moment in j-j coupling can be calculated without explicitly setting up the corresponding wave functions.

From the author's summary.

Jean, Maurice; et Touchard, Jean. Sur l'introduction de variables collectives pour la description des noyaux atomiques. C. R. Acad. Sci. Paris **245** (1957), 1001–1003.

Pauli, W. On the conservation of the lepton charge. Nuovo Cimento (10) **6** (1957), 204–215.

The discovery of the failure of parity conservation has given new interest to the search for conservation laws in weak lepton interactions, as a means for restricting the types of coupling possible. The law of conservation of leptons is as yet only tentatively established, and crucial experiments are difficult. One type of experiment is the production of a pair of negatons, without accompanying neutrinos, in a type of double β-decay. The calculation of the matrix elements for these processes may be simplified by using an invariance principle which, while not new, has been little used. The principle is "the S-matrix elements of all processes must also stay invariant (up to an arbitrary phase factor) if the initial and final states are left invariant. In this case they can therefore only depend on the invariant forms of the coupling constants". Transformations of the neutrino wave functions obviously leave invariant processes which do not contain neutrinos in the initial and final states. More generally, the transformation

$$\psi'=(\cos\alpha+i\gamma_5\sin\alpha)(a\psi+b\gamma_5\psi^c)$$

(ψ^c the charge conjugate wave function) will leave invariant matrix elements in which the distinction between neutrino and anti-neutrino does not enter explicitly. Such a transformation of the neutrino wave functions in the Hamiltonian is shown to be equivalent to a linear transformation of the coupling constants (the free Hamiltonian is invariant) and application of the above invariance principle requires the appearance only of invariant combinations of the coupling constants. The

principle of conservation of leptons then reduces the number of independent coupling constants. In particular, the two component theory reduces to that of Lee and Yang [Phys. Rev. (2) **105** (1957), 1671–1675].

C. A. Hurst (Adelaide).

Touschek, B. F. The mass of the neutrino and the non-conservation of parity. Nuovo Cimento (10) **5** (1957), 1281–1291.

It is known that if a field theory is invariant under the neutrino gauge transformation

$$\nu' = \exp(i\gamma_5\alpha)\nu, \quad \psi_i' = \exp(in_i\alpha)\psi_i,$$

where n_i is a number characteristic of the ith field, then the neutrino mass and magnetic moment must be strictly zero [Salam, Nuovo Cimento (10) **5** (1957), 299–301]. The author investigates the restriction this invariance principle imposes on the possible forms of theories involving the production or absorption of one or two neutrinos. For β-decay and π-decay, he shows that neutrino gauge invariance is incompatible with parity conservation. But for μ-decay, there are two possibilities: either S, P and T type interactions with parity not conserved, or V and A type interactions with parity possibly conserved.

Experimental evidence on the angular distribution of electrons in μ-decay seems to exclude the former type. The author claims that there is no compelling reason for abandoning parity conservation for two neutrino decays.

C. A. Hurst (Adelaide).

Matveev, A. N. The role of spin in the radiation from a "radiating" electron. Soviet Physics. JETP **4** (1957), 409–417.

L'auteur, jugeant que l'on déduit des considérations beaucoup trop générales le fait que le spin ne joue pas un rôle essentiel dans la radiation d'un électron "radiant", étudie théoriquement l'intensité, le spectre différentiel, et l'énergie totale de radiation d'un électron se mouvant dans un champ magnétique constant en tenant compte d'un spin ou d'aucun spin de l'électron. Il montre ainsi que dans le cas d'énergies qui ne sont pas trop grandes, les corrections de spin sont du deuxième ordre, alors que dans le cas des énergies relativistes extrêmes le spin change essentiellement le charactère du spectre différentiel et la grandeur de l'énergie totale irradiée.

S. Gorodetzky (Strasbourg).

Tzou, Kuo-Hsien. Champ vectoriel chargé de masse propre nulle. C. R. Acad. Sci. Paris **245** (1957), 289–291.

L'interaction d'un champ vectoriel de spin 1 avec un champ électromagnétique est formulée dans une théorie compatible, lorsque la masse propre est nulle. Le moment magnétique propre doit être deux fois celui de la théorie habituelle. *Résumé de l'auteur.*

Irving, J.; and Boucher, F. B. A variation-iteration method for short range potential problems. Proc. Phys. Soc. Sect. A. **70** (1957), 512–514.

The device of cutting off the potential at a point where the asymptotic form of the wave function is valid permits the use of a simple trial wave function in the form of a polynomial in which some of the coefficients are variation parameters. The variation-iteration method is thereby simplified. As an illustration the bound-state deuteron problem is solved for a central Yukawa interaction and the results compared with those obtained by other authors. *Author's summary.*

Tzou, Kuo-Hsien. Sur les théories matricielles du photon. C. R. Acad. Sci. Paris **245** (1957), 141–144.

Comme pour le neutrino, il y a, pour le photon (électrodynamique maxwellienne en formulation matricielle), deux théories possibles: la théorie à 6 composantes et celle à 3 composantes. Mais contrairement au cas du neutrino, ces théories du photon sont toutes deux invariantes par rapport à l'inversion de l'espace et à celle du temps. *Résumé de l'auteur.*

Kulakov, I. J. Über die Transformationsfunktionen von Teilchen mit willkürlichem Spin. C. R. Acad. Bulgare Sci. **10** (1957), 105–108. (Russian. German summary)

Es wird die Frage der Transformationsfunktion radialer Teilen der Wellenfunktion betrachtet. Es wird ein klarer Ausdruck erhalten, aus welchem folgt, daß die Transformationsfunktion nicht von Spineigenschaften der Teilchen abhängt, sondern sich nur aus ihrem orbitalen Moment l bestimmt.

Zusammenfassung des Autors.

Regge, T. On the properties of spin 2 particles. Nuovo Cimento (10) **5** (1957), 325–336.

A theory of spin 2 particles has been worked out. The wave equations are of first order and a symbolic approach which is developed permits easy computation of quantities of physical interest. An interesting application of the sum rules for spin 2 particles is the calculation of the Coulomb scattering in relativistic Born approximation. The theory is also applied to calculate the decay of a spin 2^+ particle into three pseudoscalar particles and into two Dirac particles. The bearing of the decay calculations on the spin 2^+ hypothesis for the K-meson is discussed.

Author's summary.

Szépfalusy, P. On a new exchange potential. Acta Phys. Acad. Sci. Hungar. **7** (1957), 357–364. (Russian summary)

In the investigation of systems consisting of particles of spin $\frac{1}{2}$ by the Hartree-Fock method there occurs the operator of exchange interaction. The semi-classical analogue of this operator is known for various interaction potentials. The purpose of this paper is to present a method which makes possible a more accurate determination of the semi-classical analogue of the exchange operator. The detailed calculations have been carried out for the Coulomb potential. *Author's summary.*

Konrad, Maksimilijan. Equations for the ion motion in a fixed frequency cyclotron. Glasnik Mat.-Fiz. Astr. Ser. II. **11** (1956), 253–262. (Serbo-Croatian summary)

See also: Linear Algebra: Dean. **Groups and Generalizations:** Heine. **Numerical Methods:** Metropolis. **Statistical Thermodynamics and Mechanics:** Ekstein. **Relativity:** Bergmann; Souriau.

Relativity

★ **Arzeliès, Henri. La dynamique relativiste et ses applications. Fasc I. Dynamique du point lentement accéléré. Application aux phénomènes d'interaction entre charges électriques (électromagnétisme relativiste).** Gauthier-Villars, Paris, 1957. xiv+304 pp. 4000 francs.

A textbook of Relativistic Dynamics, limited to the

Special Theory and small accelerations (for which radiation damping may be ignored), with specific application to the electromagnetic interaction of non-spinning classical point charges. First of a series: Vol. II is expected to deal with Electron Optics and Collisions, and Vol. III with Radiation Fields. Follow-up to an earlier work by the author [La cinématique relativiste, Gauthier-Villars, Paris, 1955; MR 17, 1014], to which frequent reference is made.

The author's "message" is that Electromagnetism should be derived from Relativity plus Coulomb's static law, rather than vice versa. This would imply that one deduces the Lorentz transformation from other evidence than Michelson-Morley's electromagnetic propagation experiment.

From the dynamical postulate $p=mv(1-v^2/c^2)^{-\frac{1}{2}}$ and the Lorentz transformation, Arzeliés first develops the less aesthetic features of relativity dynamics, such as longitudinal and transverse mass, transformation of the non-covariant 3-vector of force etc. The velocity dependence of transformed forces leads him to study forces of the magnetic type. The approach is non-Minkowskian (there are no space-time diagrams in the book) and the first reader is advised to omit the rather pleasanter chapters dealing with the four-dimensional representation of relativity. The author must believe in the didactic value of this program, since he is anxious to popularize relativity.

In the second part, the Biot-Savart force is obtained from the Coulomb force by a relativity transformation and electromagnetic phenomena are explained in terms of direct action. Fields are represented as a mathematical artifice, the ideas of "propagation" and even "retardation" are strongly denied, the force being static in the rest-frame of its originating charge. One is left to wonder how emission of radiation from accelerated charges (Vol. III) will fit into this scheme.

The opposition to the concept of propagation appears to have led to suppression of the Michelson-Morley background to relativity, in spite of an otherwise extensive historical and bibliographical documentation. A further conspicuous omission is Eddington's "Mathematical theory of relativity" [2nd ed., Cambridge, 1924] which does base relativity transformations on apriori reasoning and introduces electromagnetism later.

The absence of the space-time approach leads to several obscurities. The simultaneity of charge in infinitesimal volumes (XI, 2(124) e, and the related discussion II, 3 (31) a), as well as the argument against retardation (XI, 3 (132) c) suffer in this respect. Figure 28, converted into a space-time diagram, would resolve Arzeliès' conflict with the idea of retardation. There is a thorough discussion of Unipolar Induction and related phenomena, but the statement that a neutral current ring becomes charged on rotation (II, 3 (31) c) seems in error: the Special Theory does not apply.

The detailed analyses of the meaning of force, mass, energy, action and reaction, etc. are well documented. Occasional non-scientific quotations provide welcome relief. *O. Buneman* (Palo Alto, Calif.).

★**Slater, Noel B. The development and meaning of Eddington's 'Fundamental Theory'.** Cambridge University Press, New York, 1957. xii+299 pp. $7.50.

The author, in his introduction, remarks of Sir Arthur Eddington's posthumous "Fundamental theory" [Cambridge, 1946; MR 11, 144] (denoted in the following by FT) that "Many have tried to read the book; but all have found it difficult". From the existence of five or six manuscript versions of most of the book, he concludes that Eddington found it equally difficult to write. The object of the present volume is to make the reading easier by collating the existing manuscripts with each other and with the published version. The arrangement of chapters is as follows: 1. Introduction, with outline of FT (4 pages); 2. List of manuscripts with chronology; 3. Summary of FT Chapters 1–5 (35 pages); 4–8. Analysis of drafts of FT Chapters 1–5 (96 pages); 9. Summary of FT Chapters 6–8 (27 pages); 10. Summary of FT Chapters 9–11 (23 pages); 11–13. Analysis of draft of FT Chapters 6–11 (62 pages); 14. Draft of FT Chapter 13 (25 pages).

Thus the commentary, like the book, is divided into three parts: statistical theory, E-number theory, and applications (cf. the review of FT cited above). Dr. Slater has reproduced from the manuscripts "all passages which (a) contribute materially to the understanding of the final version or (b) contain developments which were later discarded but are of intrinsic interest". Unfortunately, both the final version and the earlier manuscripts are all so obscure that this higher criticism does little to clarify the meaning of Fundamental Theory, although providing much (indeed, all available) information about its development. This book is for specialists who have already struggled with the final version. {The tyro would better approach the subject through the fresh and invigorating, although often difficult, papers of Bastin and Kilmister, which Dr. Slater mentions in his Bibliography: Proc. Roy. Soc. London. Ser. A. **212** (1952); 559–576; Proc. Cambridge Philos. Soc. **50** (1954), 278–286; 439–448; **51** (1955), 454–468; Quart. J. Math. Oxford Ser. (2) **6** (1955), 161–172; [MR **14**, 227; **15**, 760; **16**, 96, 1179; **17**, 558].} *F. A. E. Pirani.*

Rosen, N. Note on relativistic motion in a scalar field. Bull. Res. Council Israel. Sect. A. **6** (1956), 55–60.

Generalizing a proposal of Marx and Szamosi [Bull. Acad. Polon. Sci. Cl. III. **2** (1954), 475–479; MR **16**, 1167], the author studies the motion of a particle of constant or variable mass moving Lorentz-invariantly in a scalar field [cf. O. Bergmann, Amer. J. Phys. **24** (1956), 38–42 for a modernized version of a similar theory due to Nordström, Ann. Physik (4) **43** (1914), 1101–1110].

F. A. E. Pirani (London).

Costa de Beauregard, Olivier. Isomorphisme de la dynamique relativiste des systèmes de points et de la statique classique des systèmes de fils. Cahiers de Phys. **11** (1957), 137–148.

On met en évidence l'isomorphisme entre la dynamique relativiste du point et la statique classique du fil, entre la dynamique relativiste des systèmes de charges électrisées de Wheeler-Feynman et la statique classique des systèmes de fils en interaction. Accessoirement, l'on discute de la définition covariante relativiste du barycentre et de l'énoncé relativiste des théorèmes généraux de la dynamique, puis de la relation entre l'irréversibilité macroscopique du rayonnement et celle de la thermodynamique. *Résumé de l'auteur.*

Mušicki, Đorđe. L'équation relativiste des ondes de matière et le principe de Pfaff. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N. S.) **9** (1956), 55–62. (Serbo-Croatian. French summary)

En partant de l'élément d'action relativiste des ondes

de matière, cet élément étant pris comme une forme de Pfaff, on déduit l'équation de Klein-Gordon des ondes de matière par l'application du principe de Pfaff dans la forme qui lui a été donnée par A. Bilimovitch [cf. MR 6, 243]. *Résumé de l'auteur.*

MacColl, L. A. Theory of the relativistic oscillator. Amer. J. Phys. **25** (1957), 535–538.

This is a study of the motions of a particle subject to the restoring force exerted by a linear spring, under the assumption that the mass depends upon the speed according to the formula given by the theory of special relativity. *Author's summary.*

García, Godofredo. Fundamentals of the special theory of relativity. Actas Acad. Ci. Lima **20** (1957), 10–58. (Spanish)

Datzeff, Assène. L'éther et la relativité restreinte. C. R. Acad. Sci. Paris **245** (1957), 827–829, 891–894.

Bergmann, Peter G. Two-component spinors in general relativity. Phys. Rev. (2) **107** (1957), 624–629.

The author develops the usual generally covariant two-component spinor theory [Infeld and van der Waerden, S.-B. Preuss. Akad. Wiss. **1933**, 380–401; cf. also W. L. Bade and H. Jehle, Rev. Mod. Phys. **25** (1953), 714–728; MR **15**, 162]. The treatment, which is elegant and concise, starts from the spin matrices rather than from the Riemannian space-time manifold, but is otherwise the usual one. *F. A. E. Pirani* (London).

Gilbert, C. The gravitational field of a star in the expanding universe. Monthly Not. Roy. Astr. Soc. **116** (1956), 678–683 (1957).

The author studies the problem of the title, previously investigated by Einstein and Straus [Rev. Mod. Phys. **17** (1945), 120–124; MR **7**, 87] and others. He compares the boundary conditions used by Einstein and Straus with those proposed by O'Brien and Synge [Comm. Dublin Inst. Adv. Studies. Ser. A. No. **9** (1952); MR **14**, 913]. *F. A. E. Pirani* (London).

Meister, H. J. Die Bewegungsgleichungen in der allgemeinen Relativitätstheorie. Ann. Physik (6) **19** (1956), 268–282 (1957).

As shown by A. Einstein, L. Infeld, and B. Hoffmann [Ann. of Math. (2) **39** (1938), 65–100], the equations of motion of the field-producing particles can be derived from the general relativity field equations of gravitation. In so doing, they considered the particles to be represented by field singularities of the spherically-symmetric Schwarzschild type. Subsequently V. Fock, A. Papapetrou and others have derived approximate equations of motion for spherically-symmetric extended bodies formed of an ideal fluid. The present paper first considers the effect of motion on an isolated fluid body, and shows that, to the approximation considered, such a body suffers the Fitzgerald-Lorentz contraction. The author then derives the equations of motion of two such bodies under their mutual gravitation, and shows that, to second order, they are the same as those derived by the previous authors, although the field itself may differ in terms of corresponding order. The agreement to terms of second order, despite the apparently different assumptions, is attributed to the orders in which the pressure and the center of mass integrals actually enter into the computation, an agreement which may not be obtained in higher order approximations.

H. P. Robertson (Pasadena, Calif.).

Buchdahl, H. A. Reciprocal static solutions of field equations involving an asymmetrical fundamental tensor. Nuovo Cimento (10) **5** (1957), 1083–1093.

The author's method of reciprocity is one of the few known methods of generating new solutions from known solutions of field equations of relativistic type. In the present paper, the above-mentioned method is developed and applied to field equations of the type used in the last form of Einstein's unified field theory. The author does not, however, restrict the dimension of his space to be four. *M. Wyman* (Edmonton, Alta.).

Petzold, Joachim. Zur Deutung fünfdimensionaler Feldgleichungen. Z. Physik **148** (1957), 192–208.

An attempt is made to give a physical interpretation in the four-dimensional world of the five-dimensional field equations. This mapping depends on five arbitrary parameters. If the mapping depends only on the coordinates of space and time, we get the projective theory of relativity. As an example, Einstein's theory is chosen. It is possible to give an interpretation of Proca's vector-meson field equations. *J. A. Schouten* (Epe).

Souriau, Jean-Marie. Equations de Dirac en schéma relativiste général. C. R. Acad. Sci. Paris **245** (1957), 496–497.

On propose, pour écrire les équations de Dirac, des variables tensorielles qui permettent l'extension de ces équations au cadre de la relativité générale, sous une forme compatible avec les axiomes que nous avons proposés dans une note antérieure. *Résumé de l'auteur.*

Souriau, Jean-Marie. Le tenseur impulsion-énergie en relativité variationnelle. C. R. Acad. Sci. Paris **245** (1957), 958–960.

Le schéma variational que nous avons proposé pour la physique relativiste permet d'établir des équations générales de conservation. Celles-ci conduisent à définir, au moyen de la présence, le tenseur impulsion-énergie de chaque phénomène physique. *Résumé de l'auteur.*

Pachner, Jaroslav. Ein Variationsprinzip für klassische Feldtheorien. Ann. Physik (6) **19** (1956), 353–368 (1957).

In a preliminary discussion of the physical background to the variational principles generally used in the derivation of field equations in the "unified" field theories, the author suggests a division of the 4-dimensional continuum such that the latter consists (1) of the vacuum and (2) of finite domains containing elementary particles. Certain qualitative arguments are presented to indicate that this does not violate the basic relativistic point of view, according to which the elementary particles cannot be of finite dimension. As a result of this construction new hypersurface integrals appear in the variational principle. A special Hamiltonian is suggested for the latter (which seems to represent a combination of the theories of Schrödinger [Proc. Roy. Irish Acad. Sect. A. **52** (1948), 1–9; MR **9**, 311] and Bonnor [Proc. Roy. Soc. London. Ser. A. **226** (1954), 366–377; MR **16**, 755]). The resulting

field equations are derived; on the basis of a method due to Papapetrou [Proc. Roy. Irish Acad. Sect. A. 52 (1948), 69–86; MR 10, 580], the spherically symmetric static solutions are discussed in detail, for these seem to exhibit some remarkable affinities with the general theory of relativity and classical electromagnetic theory.

H. Rund (Durban).

Astronomy

Ramakrishnan, Alladi, and Srinivasan, S. K. Correlation problems in the study of the brightness of the Milky Way. Astrophys. J. 123 (1955), 479–485.

Two different aspects of the paper, the mathematical and the astrophysical, require separate discussions. Mathematically, the paper is concerned with a stochastic process of the following structure. On a segment $(0, t)$ of straight line, a random number X of points are Poisson-wise distributed, with $E(X)=\lambda t$. Given $X=n>0$, the coordinates of those points are thus uniformly and independently distributed between zero and t. Let $x_1<x_2<\cdots<x_n$ be these coordinates and put $x_0=0$ and $x_{n+1}=t$. Also, let q be a random variable with an arbitrary probability density that vanishes for $q<0$ and for $q>1$. Let $q_0=1$ and $q_1, q_2, \cdots, q_n$ be n independently assumed values of q. Then, the study is concerned with the random variable

$$I(t)=\sum_{k=0}^{n}(x_{k+1}-x_k)\prod_{i=0}^{k}q_i.$$

Extending the earlier results [Ramakrishnan, Nederl. Akad. Wetensch. Proc. Ser. A. 58 (1955), 470–482, 634–645; MR 17, 637], the authors calculate product moments $E[I(t_1)I(t_2)]$ and $E(I[t_1)I(t_2)I(t_3)]$.

Astrophysically, each point x_k is identified with the intersection of a line of sight with the kth light absorbing cloud. The stars within the Milky Way located on the line of sight between the kth and the $(k+1)$st light absorbing clouds are supposed to send towards the observer the amount of light energy (measured in convenient units) equal to difference the $x_{k+1}-x_k$. Then, $I(t)$ is supposed to represent the amount of light energy reaching the observer from all the stars in the given direction up to the distance t, as transmitted by a random number X of randomly located clouds, each with a random transparency factor q. — The reviewer is uneasy about the realisticness of this mathematical model and, in particular, about the following points. (*a*) The amount of light energy reaching the observer appears independent of the distance from the stars. (*b*) The author's model is strictly one-dimensional, with the consequence that a cloud either does not intersect the line of sight or intersects it, dimming all the stars located on it, to the same degree q. On the other hand, the observations of the brightness of the Milky Way are necessarily confined to solid angles ω with vertices at the observer. Thus, particularly at great distances, a cloud may easily obscure a part of ω, not the whole. Thus, mathematically, the problem of fluctuations of brightness of the Milky Way is essentially a three-dimensional one.

J. Neyman (Berkeley, Calif.).

Nahon, Fernand. Sur une nouvelle méthode d'analyse des vitesses radiales. C. R. Acad. Sci. Paris 242 (1956), 462–464.

A new proof, based upon the characteristic function, of Ambarzumian's formula [Monthly Not. Roy. Astr. Soc. 96 (1936), 172–179], connecting the probability density of the velocity vectors of stars with the conditional density of their radial velocities, given a fixed galactic longitude α. For any components x and y of a star's velocity, the new method provides a formula for estimating the derivatives of f with respect to x or y.

J. Neyman (Berkeley, Calif.).

Jelenewskaja, N. B. Entwicklung der Störungsfunktion in eine Fourier-Reihe bezüglich der Neigung. III. Entwicklung der Störungsfunktion in eine Reihe nach den Potenzen des Exzentrizitätszuwachses. Byull. Inst. Teoret. Astr. 6 (1957), 434–465. (Russian. German summary)

Jelenewskaja, N. B. Entwicklung der Störungsfunktion in eine Fourier-Reihe bezüglich der Neigung. IV. Entwicklung der Störungsfunktion im räumlichen eingeschränkten elliptischen Dreikörperproblem und im unrestringierten n-Körperproblem. Byull. Inst. Teoret. Astr. 6 (1957), 466–486. (Russian. German summary)

García, Godofredo. On the variation of masses in the motion of planets and comets without recourse to the theory of relativity. Actas Acad. Ci. Lima 20 (1957), 59–86. (Spanish)

Zimmermann, Otto. Zur Theorie der Polarisation des Sternlichts an interstellarem Staub. Z. Naturf. 12a (1957), 647–657.

Die Absorption und Polarisation des Sternlichts beim Durchgang durch eine Wolke von teilweise ausgerichteten rotationsellipsoidischen Eisenstaubteilchen wird in 2. Näherung berechnet unter der Annahme eines einfachen, aus der Sternschnuppenstatistik entnommenen Verteilungsgesetzes für die Teilchengröße und eines Langevinschen bzw. Gausschen Verteilungsgesetzes für die Richtungen der Längsachsen. Aus den numerischen Ergebnissen kann man schließen, daß die vorliegende Näherung bis zu Teilchen von 10^{-5} cm Radius brauchbar sein wird. Eine Erweiterung der Theorie auf gröbere Teilchen wird auf anderem Wege versucht.

Aus der Zusammenfassung des Autors.

Poincelot, Paul. Propagation d'une onde électromagnétique plane à travers une couche ionosphérique. C. R. Acad. Sci. Paris 244 (1957), 3045–3047.

Eckart, Gottfried. Sur la variation de la polarisation des ondes ultracourtes due à l'hétérogénéité de la troposphère. C. R. Acad. Sci. Paris 244 (1957), 3044–3045.

Geophysics

★ **The United Kingdom contribution to the International Geophysical Year 1957-58.** The Royal Society, London, 1957. 72 pp. (1 plate). 10s.

A summary of background and plans in the various disciplines; e.g., meteorology, oceanography, seismology, etc.

Kasahara, Akira. The numerical prediction of hurricane movement with the barotropic model. J. Meteorol. 14 (1957), 386–402.

Di Benedetto, Felice. Analysis of the vorticity and divergence of the thermal wind in pure geostrophic flow. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **10** (1957), 20–28.

By applying the horizontal del operator to the thermal wind equation in vectorial form and considering only horizontal geostrophic motion, expressions are found for the vorticity and divergence of thermal wind. It is shown that vorticity, in the case of pure geostrophic flow at all levels, depends upon four factors, namely: latitude, horizontal distribution of temperature, temperature lapse rate and divergence of horizontal temperature gradient. The divergence of the thermal wind, on the other hand, depends only upon the orientation of the mean isotherms of an isobaric layer relative to the parallel circles. These terms are evaluated and it is shown that the latitude term and the lapse rate term are far greater than the remaining terms of the vorticity equation for geostrophic flow. A criterion is then given which enables one to determine the conditions for irrotational horizontal motion with vertical geostrophic wind shear. *Author's summary.*

Frenkiel, J.; and Zacks, S. Wind-produced energy and its relation to wind regime. Bull. Res. Council Israel. Sect. A. **6** (1957), 189–214.

Three methods of estimating wind-produced energy from known wind regime and assumed characteristics of two types of aerogenerators are described: (1) based on the known frequency distribution of hourly mean wind speeds; (2) based on the formula of the power in the wind as a function of cumulative frequency; (3) graphical method. *From the authors' summary.*

Malinovskaya, L. N. On a method of calculating the dynamic characteristics of seismic waves. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 426–439. (Russian)

Adachi, Ryuzo, A problem in seismic prospecting. Kumamoto J. Sci. Ser. A. **2** (1956), 363–377.

Coulomb, J. Sur une origine aurorale possible de certaines pulsations géomagnétiques. Ann. Géophys. **13** (1957), 91–102.

There is a priori possibility for some pulsations to be generated during polar aurorae and to be propagated from polar zones towards lower latitudes. A theory, rather incomplete mathematically, of the propagation, gives correct orders of magnitude for "giant pulsations". But for the "pulsation trains" often associated with geomagnetic bays, the results are in contradiction with observated phenomena. *Author's summary.*

OTHER APPLICATIONS

Economics, Management Science

Rosenblatt, David. On linear models and the graphs of Minkowski-Leontief matrices. Econometrica **25** (1957), 325–338.

A finite square matrix $A=(a_{ij})$ is said to be of Minkowski-Leontief type if its elements are nonnegative and no row-sum exceeds unity. Associated with each such matrix is a directed graph G whose vertices α_j correspond to the rows of A and whose edge (α_i, α_j) exists if and only if $a_{ij}>0$. A cyclic net is a subgraph H of G such that each vertex of H is connected (directly or via a sequence of edges of H) to every vertex of H. H is closed if it is maximal in G and if every point of G connected to any point of H is in H. The cyclic net notion is clearly related to the concept of indecomposability of nonnegative square matrices, particularly of stochastic matrices, i.e., those whose rows each sum to unity. The main problem considered is the existence and properties of $(I-A)^{-1}$. Necessary and sufficient for the existence of $(I-A)^{-1}$ is that G contain no cyclic net or that each closed cyclic net of G should contain a vertex associated with a row of A whose sum is less than one (i.e., that any closed sub-economy should contain one industry with positive "leakage"). If $(I-A)^{-1}$ exists, w is any nonnegative nonnull vector, and x is the solution of $x(I-A)=w$, then $x_j>0$ (x is obviously nonnegative) if and only if $w_j>0$ or there exists a $w_k>0$ for which α_j is connected to α_k. Further similar results are obtained by embedding A in a stochastic matrix and using the theory of finite Markoff chains. *R. Solow.*

★**Giaccardi, Fernando. Considerazioni su alcune disuguaglianze e applicazioni.** Scritti matematici in onore di Filippo Sibirani, pp. 123–141. Cesare Zuffi, Bologna, 1957.

Sono considerate alcune diseguaglianze numeriche che generalizzano quelle di Liapounov e di Jensene se ne traggono applicazioni a questioni di matematica finanziaria. *G. Fichera* (Roma).

Strotz, Robert H. The empirical implications of a utility tree. Econometrica **25** (1957), 269–280.

Given a consumer's preference field with commodities q_j $(j=1, \cdots, n)$, these are assumed to form m bundles $q_{i1}, \cdots, q_{ic_i}$ $(i=1, \cdots, m; c_1+\cdots+c_m=n)$ such that the preference function can be written $U[q_1, \cdots, q_n]=U^*[V_1(q_{11}, \cdots, q_{1c_1}), \cdots, V_m(q_{m1}, \cdots q_{mc_m})]$, called a utility tree with m branches. Theorem: Writing Y for consumer income, p_k for the price of q_k, and $q_j=a_{j0}+\sum_{k=1}^{n} a_{jk}p_k+b_jY$ for the tangent plane of the demand function of q_j, suppose q_r, q_s belong to one branch and q_t to another; then $a_{rt}/a_{st}=b_r/b_s$. Corollaries on price index construction and consumer expenditure allocation are developed. The comments on other work are somewhat sweeping. {The Paerto-Slutsky theory contains three or four theorems of fundamental relevance for empirical demand analysis, and along this line the author brings important new material, whereas the implications in welfare economics remain "frankly disappointing in their paucity".} *H. Wold* (Uppsala).

Mills, Edwin S. The theory of inventory decisions. Econometrica **25** (1957), 222–238.

A firm makes, stocks, and sells a single commodity, whose market price is constant. Unit production cost (charged against sales, not output) is constant. Carrying cost per unit of stock is constant. The penalty for inability to fill orders is proportional to the volume of unfilled orders. There is also a cost which is proportional to the absolute change in output from the previous period. Thus, if x, y, z represent demand, total supply (output plus opening inventory), and output, and if the constants appearing above are, in order, p, c, r, a, b, then profits in any period are $(p-c)x-r(y-x)-b|\Delta z|$, or $(p-c)y-a(x-y)-b|\Delta z|$, respectively, for x less than or greater than

y. The demand per period is in part uncertain, and the firm must make output decisions. The consequences of two decision criteria, maximum expected profit and minimax regret, are worked out and the corresponding decision rules are exhibited. There is an econometric application to data for six English manufacturing firms. The minimax regret model comes out slightly better than the expected profit model, but there is a strong tendency for current output to have a negative partial regression on lagged output, which seems inconsistent with the cost attached to changing production. *R. Solow.*

Verdoorn, P. J. Complementarity and long-range projections. Econometrica **24** (1956), 429–450.

Long-term forecasts are discussed with special reference to the nature of the relation between factors of production. It is argued that complementarity is a more suitable assumption than substitution, even though the former is more likely to lead to disequilibrium with the implied necessity of institutional price fixing. A particular model, used by the Netherlands Central Planning Bureau, is presented. This model, which gives a prominent part to foreign trade, requires the introduction of certain policy variables to maintain equilibrium. Little attention is paid to possibilities of substitution through reactions on the demand side. *H. S. Houthakker* (Cambridge, Mass.).

Rufener, Ernst. Sterbegesetze, für welche gewisse Lidstonesche Näherungen exakt erfüllt sind. Bl. Deutsch. Ges. Versicherungsmath. **3** (1957), 163–170.

If $\bar{a}_{x_1x_2\cdots x_m}=\int_0^\infty(1+i)^{-t}\prod_{j=1}^m l_{x_j}^{-1}l_{x_j+t}dt<\infty$, the equation

$$\bar{a}^{-1}_{x_1x_2\cdots x_m}=m^{-1}\sum_{j=1}^m \bar{a}^{-1}_{x_jx_j\cdots(m)}$$

implies that $l_x=ks^xg^{c^x}$ ($c\geqq 1$, $s\geqq 1$, $0<g\leqq 1$, $k>0$), where $(1+i)c=s^m$. If the upper limit of the integral is replaced by $n<\infty$ the parameter c becomes identically equal to unity. *H. L. Seal* (New York, N.Y.).

Zwinggi, Ernst. Ansätze für die Gewinnermittlung nach der kontinuierlichen Methode. Mitt. Verein. Schweiz. Versich. Math. **57** (1957), 45–53.

An example is given of how the surplus in life insurance may be divided into its components (interest, mortality, and loading surpluses) when the premium calculations are based on integration formulae (as opposed to summation formulae). *K. Medin* (Stockholm).

Vegas, Angel. Statistical inference in biometric models and its application to life insurance. Trabajos Estadist. **7** (1956), 263–286. (Spanish. English summary)

The paper gives a statistical approach to the mathematics of the mortality functions by studying the distribution of the age of death of an observed group of mutually independent lives following the same mortality law. Estimates of the parameters of Makeham's and other mortality formulae are derived by orthogonal polynomes. The variance of a company's gain is derived in accordance with the classical individual risk theory. *P. Johansen.*

Dienst, Hans-Rudolf. Witwenrentenanwartschaft und Ehestandshäufigkeit. I, II. Bl. Deutsch. Ges. Versicherungsmath. **2** (1956), 413–430; **3** (1957), 171–181.

Various expressions for the capital value of a collective widow pension are given and discussed. The formulae, however, may be somewhat simplified by using the continuous method. *P. Johansen* (Copenhagen).

See also: Probability: Runnenburg. **Programming, Resource Allocation, Games:** Gale; Tucker; Goldman; Thompson.

Programming, Resource Allocation, Games

★ **Symonds, Gifford H. Linear programming: the solution of refinery problems.** Esso Standard Oil Co., New York, N.Y., 1955. ii+74 pp.

Several linear programs are formulated and solved in a nontechnical introduction to some industrial uses and computational procedures of linear programming. The characteristic "tableau" and "change of basis" of the Simplex Method are described, but no justification for the complete method is given. Two of the examples used are based on "constrained" matrix games (a decision maker against nature), while a nonzero sum two person game is explored in another. *A. W. Tucker.*

Tucker, A. W. Linear and nonlinear programming. Operations Res. **5** (1957), 244–257.

This paper is the first of four from the "Symposium on Modern Techniques for Extremum Problems" held at the meeting of the Operations Research Society of America at Los Angeles, August, 1955. Like the two succeeding papers, it reviews important recent developments in the theory of nonlinear "programming problems", problems of the extremization of functions of variables subject to constraints.

The present paper consists principally of an exposition of results of H. W. Kuhn and the author [Proc. 2nd Berkeley Symposium Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 481–492; MR 13, 855] on saddlepoint theorems and the use of Lagrange multipliers in nonlinear programming problems. The central result is the equivalence of the problems: (i) To find $x=(x_1, \cdots, x_q, \cdots, x_n)$ minimizing the real function $f(x)$ such that $x_j\geqq 0$ ($j=1, \cdots, q$) and $g_i(x)\geqq 0$ ($i=1, \cdots, p$), $g_i(x)=0$ ($i=p+1, \cdots, n$), f being supposed convex and the g_i concave; (ii) to find x and $u=(u_1, \cdots, u_m)$ solving the saddlepoint problem

$$\min_{(x|x_j\geq 0,\ j=1,\cdots,q)}\ \max_{(u|u_i\geq 0,\ i=1,\cdots,p)}\ [f(x)+\sum_{i=1}^m u_ig_i(x)].$$

The specialization of this result to the duality theorem of linear programming in the case that f and the g_i are linear is given. It is illustratively applied to two "network" problems: The Hitchcock-Koopmans problems of minimizing the cost of transporting assigned quantities of an homogeneous commodity from several sources to several destinations over a given network of routes (an instance of linear programming), and the classical Kirchhoff-Maxwell problem of determining the distribution of current in an electrical network by minimizing power loss (an instance of "quadratic" programming). *P. Wolfe* (Santa Monica, Calif.).

Arrow, Kenneth; and Hurwicz, Leonid. Gradient methods for constrained maxima. Operations Res. **5** (1957), 258–265.

This second paper from the "Symposium on Extremum Problems" reviews certain proposals for computing solutions of the nonlinear programming problem $\operatorname{Max}\{f(x)|\ x_i\geqq 0\ \ (i=1, \cdots, m),\ \ g_j(x)\geqq 0\ \ (j=1, \cdots, n)\}$. Relying on results described in the paper reviewed above, a solution of this problem can be given as part of a saddlepoint of the "modified" Langrangian $\varphi(x_1, \cdots, x_m; y_1, \cdots, y_n)=f(x)+\sum_{j=1}^m y_j\psi[g_j(x)]$ for suitable functions ψ. As a method for finding a saddlepoint, the authors consider the differential equations

$$dx_i/dt=\{\partial\varphi/\partial x_i \text{ if } \partial\varphi/\partial x_i>0 \text{ or } x_i>0, \text{ otherwise } 0\},$$
$$du_j/dt=\{-\partial\varphi/\partial u_j \text{ if } \partial\varphi/\partial u_j<0 \text{ or } u_j>0, \text{ otherwise } 0\}.$$

The results announced are: (i) If ψ is the identity, f is strictly concave, and the g_j are concave, then solutions to these equations from any nonnegative starting values exist and convergence to a saddlepoint of φ (due, excepting existence of solutions, to T. Kose [Econometrica 24 (1956), 59–70; MR 17, 1105]; completed by H. Uzawa [Dept. of Econ., Stanford Univ., Tech. Rep. no. 36]). (ii) Let $\psi(t)=1-(1-t)^{1+\eta}$ $(\eta>0)$. If the programming problem has a solution, then there exists y^0 such that (x^0, y^0) is a saddlepoint of φ in the product of $\{y \mid y\geqq 0\}$ with some neighborhood of x^0 [Arrow and Hurwicz, Proc. 3rd Berkeley Symposium Math. Statist. and Probability, 1954–1955, v. 5, Univ. of California Press, 1956, pp. 1–20; MR 18, 954]. (iii) if f and the g_j are linear, $\psi(t)=1-\exp(-\eta t)$, and the (linear) programming problem has solutions, then solutions to the differential equations exist for any nonnegative starting values which converge to a saddlepoint of φ. *P. Wolfe.*

Dantzig, George B. Discrete-variable extremum problems. Operations Res. **5** (1957), 266–277.

The third paper of the symposium on extremum problems deals with the application of linear programming methods to problems of finding the extremum of a function whose variables are restricted to discrete domains. Instances of such problems are those described adjectivally as 'assignment,' 'traveling-salesman,' 'flight-scheduling,' etc.; their recasting for linear programming results generally in the computational problem: Minimize $\sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij}x_j=b_i$ $(i=1, \cdots, m)$ and $x_j=0$ or 1 $(j=1, \cdots, n)$. (This recasting is illustrated for "the marriage" [assignment] problem.) If the constraints $x_j=0$ or 1 are replaced by $0\leqq x_j\leqq 1$ $(j=1, \cdots, n)$, the resulting extremum problem is amenable to solution by linear programming methods; and in certain cases — notably the assignment problem, which can be stated as finding that permutation of the rows of a matrix which minimizes its trace — the solution of the modified problem will solve the original. This is also the case for the problem of finding the shortest route between two points in a graph whose links are of given lengths; such a problem is exhibited and solved. This is not the case for the 'traveling-salesman' problem of finding the shortest cyclic chain of links passing through all the nodes of a graph, but particular large problems have been solved by finding for each case a small number of equations which, when added to the above, restore the commonness of solutions [Dantzig, Fulkerson, and Johnson, J. Operations Res. Soc. Amer. **2** (1954), 393–410; MR **17**, 58]. This technique is illustrated on the 'knapsack' problem of maximizing one linear form of several (0 or 1) variables while requiring that another linear form not exceed a given bound. *P. Wolfe* (Santa Monica, Calif.).

Bellman, Richard. Dynamic programming and the numerical solution of variational problems. Operations Res. **5** (1957), 277–288.

The purpose of this paper is to illustrate the application of the theory of dynamic programming to the numerical solution of a large class of variational problems of the type occurring in a variety of applications. Our aim is to present a simple, readily applicable technique, requiring no mathematical background beyond elementary calculus, which can be used to compute the solution of a variety of problems in a routine fashion, with no regard to linear or nonlinear, stochastic or deterministic features of the underlying processes. [From the author's preface.] *P. Wolfe* (Santa Monica, Calif.).

Hildreth, Clifford. A quadratic programming procedure. Naval Res. Logist. Quart. **4** (1957), 79–85.

A procedure, together with several possible variants, is described for the following programming problem: $\max \theta(x)=x'Cx+d'x$, subject to $Gx\geqq h$, where x is an n by 1 (column) vector whose elements are real variables, C is a negative definite matrix, G is an m by n matrix, d' is 1 by n, and h is m by 1. The elements of C, G, d' and h are known constants.

Beale, E. M. L. On minimizing a convex function subject to linear inequalities. (Symposium on linear programming.) J. Roy. Statist. Soc. Ser. B. **17** (1955), 173–184; discussion, 194–203.

The minimization of a convex function of variables subject to linear inequalities is discussed briefly in general terms. Dantzig's Simplex Method is extended to yield finite algorithms for minimizing either a convex quadratic function or the sum of the t largest of a set of linear functions, and the solution of a generalization of the latter problem is indicated. In the last two sections a form of linear programming with random variables as coefficients is described, and shown to involve the minimization of a convex function. *Author's summary.*

Frank, Marguerite; and Wolfe, Philip. An algorithm for quadratic programming. Naval Res. Logist. Quart. **3** (1956), 95–110.

A finite iteration method for calculating the solution of quadratic programming problems is described. Extensions to more general non-linear problems are suggested. *Authors' summary.*

Bellman, Richard. A variational problem with constraints in dynamic programming. J. Soc. Indust. Appl. Math. **4** (1956), 48–61.

In another article [Bellman, R.; Fleming, W. H.; and Widder, D. V. Ann. Mat. Pura Appl. (4) **41** (1956), 301–323; MR **18**, 51] the problem was considered of maximizing the functional $J(y)=\int_0^T F(x, y)dt$ subject to the constraints $dx/dt=G(x, y)$, $x(0)=c$, $0\leqq y\leqq x$, using the classical techniques of the calculus of variations, with modifications imposed by the restraint $0\leqq y\leqq x$.

In the present article we treat the problem using the approach of the theory of dynamic programming. The basic idea is to consider a variational problem as a multistage decision process of continuous type. We shall show how we may obtain a functional equation for $U(c, T)=\max_y J(y)$ and how a limiting form of this functional equation may be used to determine the structure of the extremal curve given certain simple structural properties of $F(x, y)$ and $G(x, y)$. *From the introduction.*

Bellman, Richard. Notes on the theory of dynamic programming. IV. Maximization over discrete sets. Naval Res. Logist. Quart. **3** (1956), 67–70.

The theory of dynamic programming is applied to a class of problems involving maximization over discrete sets. The solution is made to depend on the solution of a class of functional equations. [cf. Bellman, Bull. Amer. Math. Soc. **60** (1954), 503–515; MR **16**, 732.] *Author's summary.*

Ford, L. R., Jr.; and Fulkerson, D. R. A primal-dual algorithm for the capacitated Hitchcock problem. Naval Res. Logist. Quart. **4** (1957), 47–54.

Koopmans, Tjalling C.; and Beckmann, Martin. Assignment problems and the location of economic activities. Econometrica **25** (1957), 53–76.

Two problems in the allocation of indivisible resources

are discussed. Both can be interpreted as problems of assigning plants to locations. The first problem, in which cost of transportation between plants is ignored, is found to be a linear programming problem, with which is associated a system of rents that sustains an optimal assignment. The recognition of cost of interplant transportation in the second problem introduces complications which call for more laborious and largely unexplored computations and which also appear to defeat the price system as a means of sustaining an optimal assignment. *Authors' summary.*

Jewell, Wm. S. Warehousing and distribution of a seasonal product. Naval Res. Logist. Quart. **4** (1957), 29–34.

A problem of minimizing linear storage and transportation cost for the shipment of an homogeneous commodity available, during T time periods, at m sources in different quantities than required at n sinks is formulated as an mT-source, nT-sink transportation problem. Solutions of the problem obtained on Whirlwind I up to $mT=60$, $nT=291$ are described. *P. Wolfe.*

Williams, Noel. The allocation of scarce materials between products. Appl. Statist. **5** (1956), 166–176.

The principles of linear programming are outlined and it is shown, by a simplified example worked out in detail, how linear programming can be applied to problems of material allocation. *Author's summary.*

Smith, Wayne E. Various optimizers for single-stage production. Naval Res. Logist. Quart. **3** (1956), 59–66.

This note develops theorems which enable one to schedule operations on a single-stage production system so as to minimize certain functions or optimizers. It is shown that the maximum tardiness of job completion is minimized by ordering the jobs according to the date the job is due to be completed. The sum of times until each job is completed may be minimized under various conditions. If all jobs can be completed by their due dates, a weighted sum of completion times can be minimized, subject to all jobs being completed by their due dates. This is done by scheduling last the jobs with the largest ratio of processing time to weighting factor among those jobs which will be completed on time if scheduled last; the next to last job is scheduled by applying the same rule to the remaining jobs, and so forth, until the complete ordering is obtained. *Author's summary.*

Gale, David. The basic theorems of real linear equations, inequalities, linear programming, and game theory. Naval Res. Logist. Quart. **3** (1956), 193–200 (1957).

The purpose of this note is to present short and direct proofs of five important theorems of real linear algebra, three existence theorems concerning solutions of linear equations and inequalities, and the basic theorems of linear programming and game theory. The theorems are not new but have recently assumed new importance because of their application to economics, engineering, logistics etc. *From the introduction.*

★ **Gale, David. The theory of matrix games and linear economic models.** Department of Mathematics, Brown University, Providence, R. I., June, 1957. Prepared under contract with the Office of Naval Research, Contract Nonr-562(15). xiii+265 pp. (mimeographed)

"These notes are based on lectures given as part of a graduate course in the Mathematics Department of Brown University... Their purpose is to give a unified mathematical treatment of ... linear economic models." The result is an extraordinarily neat and compact survey of zero-sum two-person games, linear programming, pure exchange models (and, implicitly, the elementary properties of finite Markoff chains), and pure production models of the Leontief and von Neumann types. Economic interpretations are usually given, but on the whole the treatment is abstract. Although the book is in principle self-contained, the author wisely recommends that it ought not to be used as a text at any lower level than graduate mathematics. On the other hand, some mathematical maturity could be replaced by economic maturity, and these notes should be of real interest to economic theorists.

The distinctive unity of the notes rests on Chapter III which provides the mathematical scaffolding. There the duality theorems of vector spaces are extended to include inequalities [see the paper reviewed above and, also, the two papers reviewed below]. The results are presented purely algebraically as "alternative theorems"; for example the classic separation theorem for polyhedral cones appears as: "$xA=a$ has no nonnegative solution if and only if $Ay\leqq 0$ and $ay>0$ has a solution". A discussion in terms of convex polyhedral cones and their polars then follows, and the connection is briefly pointed out. In economics the algebraic theorems are made to do the work.

Chapters I and II introduce the subject with a heuristic discussion of zero-sum two-person games and some preliminary theory to reduce the general case to the symmetric one. Chapter IV, armed with the algebra of Chapter III, proves the Fundamental Theorem and then goes on to discuss the structure of the convex polytopes of optimal strategies for the two players, and to sketch the construction of a game whose solutions consist of any given pair of convex polytopes satisfying the structure conditions.

Chapter V is on linear programming, and covers the fundamental duality and existence theorems, the theorem on basic solutions, and the equivalence of games and linear programs. More care than usual in expositions is given to mixed equality and inequality constraints. There is a clear exposition of the simplex method. Chapter VI is devoted to transportation problems and integral linear programming, based on the network flow theory of Ford and Fulkerson and the work of Kuhn on the assignment problem. The "Max flow min cut" theorem is proved and used to give efficient computational methods. Many examples are worked.

Chapter VII is on exchange models; e.g., n countries trade with each other and country j spends the nonnegative fraction β_{ij} of its income on imports from country i. If $B=(\beta_{ij})$ and y is the vector of incomes, the latter satisfies $By=y$ in the static theory. Only the "Say's Law" case where the columns of B each sum to one is discussed, so the theory is identically that of finite Markoff chains. Equilibrium states and convergence are briefly studied and some of the classical structure theorems deduced. A noteworthy pedagogical fact is that no use is made of characteristic roots.

The final chapter is a condensed and elegant treatment of linear models of production; closed and open Leontief models (including the Samuelson-Arrow nonsubstitution theorem, very neatly proved), the efficient-point analysis of Koopmans, and the expanding model of von Neumann. The treatment of the latter is especially perspicuous. Some new results are given. *R. Solow.*

★**Tucker, A. W. Dual systems of homogeneous linear relations.** Linear inequalities and related systems, pp. 3–18. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

For antisymmetric $M=(m_{ik})$, the system (*) $Mx\geqq 0$, $x\geqq 0$ has solutions with $Mx+x>0$ and, for every i, exactly one of $\sum m_{ik}x_k\geqq 0$ and $x_i\geqq 0$ is slack (i.e., is fulfilled with $>$ sign for some solution) in (*). Similar statements for the system $-Ax-By=0$, $-Cx-Dy\geqq 0$, $x\geqq 0$, $v\geqq 0$, $uA+vC\geqq 0$, $uB+vD=0$ and its subcases imply, and provide elegant schematic formulations for, transposition theorems of Farkas, Gordan, Stiemke, von Neumann and the reviewer. *T. S. Motzkin.*

★**Goldman, A. J. Resolution and separation theorems for polyhedral convex sets.** Linear inequalities and related systems, pp. 41–51. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

From the results of the paper reviewed above, the author derives variants of theorems by the reviewer. Resolution means representation as convex sum, in this case of a polyhedron and a cone. To the references, add Burger, Z. Angew. Math. Mech. **36** (1956), 135–139 [MR **18**, 417]. *T. S. Motzkin* (Los Angeles, Calif.).

★**Thompson, Gerald L. On the solution of a game-theoretic problem.** Linear inequalities and related systems, pp. 275–284. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

A model of an expanding economy is reduced to two m by n matrices, A, B, with non-negative entries, such that value $(-A)<0<$value (B). The problem is to find a convex combination $\rho(-A)+(1-\rho)B$ with (i) value 0, such that (ii) for some optimal strategies x, y, $xBy>0$. Here ρ determines the expansion rate, (i) is the criterion of stability, and as for (ii), xBy is the total price of the goods produced. The set of all ρ in [0, 1] satisfying (i) is a non-empty closed interval, by continuity. Theorem 1: Both endpoints of this interval satisfy (ii) also. Theorem 2: There are only finitely many values of ρ satisfying (i) and (ii). Theorem 2 is obtained by associating to each ρ a row i and column j such that $x_i b_{ij} y_j>0$; these are shown to be different for different ρ, so that min (m, n) is a bound for the number of solutions. However, a counterexample shows that there may be more than one ρ, i.e., more than one possible expansion rate, without a complete decomposition of the economy. Considerable use is made of the concept of decomposability and of results of Tucker [see the paper reviewed second above]. The paper concludes with a theorem facilitating solution of games which have decomposable matrices. *J. Isbell.*

Berge, Claude. La fonction de Grundy d'un graphe infini. C. R. Acad. Sci. Paris **242** (1956), 1404–1407.

En vue de différentes applications à la théorie des jeux, on définit, pour des graphes orientés quelconques, une notion introduite par P. M. Grundy pour les jeux de Nim finis. [Eureka **2** (1939), 6–8; voir aussi l'oeuvre analysée ci-dessous]. En outre, il est donné ici une nouvelle généralisation du théorème de Zermelo-von Neumann pour des jeux infinis (avec information complète). [Cf. mêmes C. R. **241** (1955), 455–457; MR **17**, 1223.]

Résumé de l'auteur.

Berge, Claude; et Schützenberger, Marcel Paul. Jeux de Nim et solutions. C. R. Acad. Sci. Paris **242** (1956), 1672–1674.

The game of Nim is defined as a game for two players in a generalized form with full information without go-between, in which there exist positions K and L such that

$$K\cap L=\Lambda,\ K\cup L=X_0=\{x:\Gamma x=\Lambda\},$$
$$(K.1)\cup(L.2)=K_1,\ (L.1)\cup(K.2)=K_2,$$

where $x.1$ is a position in which it is the first player's turn to more; $\Gamma x.2$ is the set of alternatives at x, and K_i is a winning position for the ith player. Such a game is denoted by (Γ, K, L). The function $g(x)$, equal to zero when $x\in L$, equal to unity when $x\in K$ and equal to $\min_{k=0,1,2}(k\notin\{g(y):y\in\Gamma x\})$ is called Grundy's function for the game (Γ, K, L). (cf. the article reviewed above).

If for the game (Γ, K, L) there exists a Grundy function $g(x)$ and if the play has reached a position $x.2$ for which $g(x)=0$, then the first player does not lose.

Let us suppose $\Gamma^+(A)=\{x:\Gamma x\neq\Lambda,\ \Gamma x\subset A\}$. Let the game (Γ, K, L) be such that $\Gamma^+\{x:g(x)\neq 0, 1\}=\Lambda$. The Grundy function for the game (Γ, L, K) is equal to zero when $g(x)=1$, equal to unity when $g(x)=0$, and equal to $g(x)$ in all other cases.

The concept is introduced of a solution relative to A of an n-person game in normal form (A is a subset of the set of imputations of the game) having a solution in the sense of Neumann-Morgenstern for $A=\Lambda$. On the basis of the theory of the game of Nim statements are formulated about the existence and uniqueness of such solutions.

I. V. Romanovskiĭ (RŽMat 1957, no. 714).

See also: Integral and Integrodifferential Equations: Bellman. **Numerical Methods:** Jacobs; Greenwald. **Statistics:** Blackwell and Hodges.

Biology and Sociology

Haskey, H. W. Stochastic cross-infection between two otherwise isolated groups. Biometrika **44** (1957), 193–204.

The Bailey stochastic model [N. T. J. Bailey, Biometrika **37** (1950); 193–202, MR **12**, 431] for the spread of infection during an epidemic in an isolated group is generalized as follows: Two groups are considered with sizes n_1 and n_2, h infectives being then introduced into the first group. The four possible probabilities of infection of an individual of one group by a member of his own or the other group, during time dt, are assumed proportional to the numbers infected and uninfected in the infector's group, and proportional to dt. The special case $n_1=2$, $n_2=3$, $h=1$ is handled rather completely. For the case of general n_1, n_2, h, heuristic considerations, involving the assumption that infectives travel back and forth, spending equal time in the two groups, lead to results in closed form for the mean numbers of susceptibles in the two groups as a function of t. The paper contains no summary. *R. F. Tate* (Seattle, Wash.).

Patlak, Clifford S. Contributions to the theory of active transport. II. The gate type non-carrier mechanism and generalizations concerning tracer flow, efficiency, and measurement of energy expenditure. Bull. Math. Biophys. **19** (1957), 209–235.

Suppose that a particle passes through a very thin barrier or membrane by reacting with one side of the membrane to form a complex which then reorganizes so as to liberate the particle on the other side of the membrane; such a mechanism, which leaves the membrane in its original condition, is called a gate type non-carrier mechanism. The author discusses the rate of transport through such a gate type mechanism, as

well as the efficiency of the mechanism. He concludes with a discussion of generalizations which may hold for other transport models, in addition to the gate type model, namely, the dependence of transport across the membrane on the concentration on the "final" side of the membrane, the possibility of very high efficiencies, and experimental methods for determining the work done by the system in performing the active transport.
R. G. Stanton (Waterloo, Ont.).

★ **Kempthorne, Oscar. An introduction to genetic statistics.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1957. xvii+545 pp. $12.75.

This book is the first one to cover systematically all the principal results of the statistical theory of genetics: segregation ratios, large and small natural and artificial populations, correlations, etc. It includes some new results due to the author. Adequate bibliographies cover points where space does not allow a full discussion.
C. A. B. Smith (London).

Landahl, H. D. Population growth under the influence of random dispersal. Bull. Math. Biophys. **19** (1957), 171–186.

A population is considered which grows according to the logistic equation while spreading out at random. An approximate method is used to obtain transient and steady-state values for various simple boundary conditions such as that of a population started in an infinite one- or two-dimensional region with or without reflecting or absorbing barriers. An approximate steady-state solution is given for the one-dimensional case of two neighboring regions having different growth rates, mobilities, and degrees of attractiveness. (From the author's summary.)
J. L. Snell (Hanover, N.H.).

See also: Statistics: Diederich, Messick and Tucker. **Economics, Management Science:** Vegas.

Information and Communication Theory

Kochen, Manfred. An information-theoretical model of organizations. Trans. I.R.E. **PGIT-4** (1954), 67–75.

A set of axioms are given for describing an organization, characterized by imperfect communication among its members, the degree of imperfection being measured in information-theoretical terms. A number of examples are given. No reference is made to somewhat similar formulations of Marschak [see Thrall, Coombs, and Davis, Decision processes, Wiley, New York, 1954; MR **16**, 605].
K. J. Arrow (Stanford, Calif.).

Stam, A. J. On measures of information. Nederl. Akad. Wetensch. Proc. Ser. B. **60** (1957), 201–211.

Generalizations proposed by M. P. Schützenberger [C. R. Acad. Sci. Paris **232** (1951), 925–927; MR **12**, 623] and S. Kulback and R. A. Leibler [Ann. Math. Statist. **22** (1951), 79–86; MR **12**, 623] are combined. A measure of information is called "consequent" when the passage from discrete to continuous information does not give rise to an infinite term, as it does in the case of the Wiener-Shannon measure. Fischer's measure of information, as well as the Kulback-Leibler generalization of the Wiener-Shannon measure are shown to be "consequent".
S. K. Zaremba (Wolverhampton).

Golay, Marcel J. E. Binary coding. Trans. I.R.E. **PGIT-4** (1954), 23–28.

This paper examines lossless binary coding and shows that such codes can exist only for lengths which have a finite upper bound. The author discusses both the usual symbol correcting codes as well as message correcting codes which are more general. *R. W. Hamming.*

Bloom, F. J.; Chang, S. S. L.; Harris, B.; Hauptschein, A.; and Morgan, K. C. Improvement of binary transmission by null-zone reception. Proc. I.R.E. **45** (1957), 963–975.

When information is transmitted in a binary code the customary receiver prints either a 0 or a 1 for each received pulse. This paper discusses the effect of using 3 or 4 levels of detection, and refers to the extra levels as 'null-zones'. The use of these null-zones significantly improves the information rate for both straight coding and simple error detecting codes. The effects of different noises and of errors in setting the edges of the null-zones are discussed. *R. W. Hamming* (Murray Hill, N.J.).

Silverman, Richard A.; and Balser, Martin. Coding for constant-data-rate systems. Trans. I.R.E. **PGIT-4** (1954), 50–63.

This paper is a condensation of a couple of papers by the same authors on the "Wagner codes". The basic Wagner code, unlike the Hamming code, was designed to operate at a constant data rate. It amounts to recording the received pulse heights of a single error detecting code, and in case a single error is indicated, changing the pulse that is most likely to be the wrong one. Extensions to Hamming-Wagner codes are discussed and evaluated, along with the Reed-Muller codes, as well as variations of these codes. The noise is assumed to be "white Gaussian noise." *R. W. Hamming.*

Blanc-Lapierre, André; Dumontet, Pierre; et Savelli, Michel. Sur quelques points de la théorie de la détection quadratique du bruit de fond. C. R. Acad. Sci. Paris **242** (1956), 2911–2913.

"The authors study the spectrum of $X^2(t)+S^2(t)$, where $X(t)$ is a random, stationary, Laplacian function and $S(t)$ is its transform by some specific linear filters." (Authors' summary.) *R. A. Leibler* (Silver Spring, Md.).

Hirschman, I. I., Jr. A note on entropy. Amer. J. Math. **79** (1957), 152–156.

For $f(x) \in L^2(-\infty, \infty)$ and $g(y) \sim \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx$ and $f(x) \sim \int_{-\infty}^{\infty} g(y)e^{2\pi ixy}dy$, where "$\sim$" refers to convergence in the mean of L^2, the author proves: If $\|f\|=\|g\|=1$, then $E[|f(x)|^2]+E[|g(y)|^2] \leq 0$, whenever the left side has meaning, where $E[\phi(x)]=\int_{-\infty}^{\infty} \phi(x) \log \phi(x)dx$. He conjectures that $E[|f(x)|^2]+E[|g(y)^2] \leq \log 2-1$ may hold under the same conditions. *R. A. Leibler.*

Rothstein, Jerome. Information, organization and systems. Trans. I.R.E. **PGIT-4** (1954), 64–66.

Organization is defined as a generalization of the notion of channel capacity of communication theory. It applies to systems of variables related by a joint probability distribution and measures the degree of correlation between the variables. The concept may be applied to such diverse systems as information storage devices, production techniques, and electrical circuits.
D. E. Muller (Urbana, Ill.).

Van Meter, David; and Middleton, David. Modern statistical approaches to reception in communication theory. Trans. I.R.E. **PGIT-4** (1954), 119–145.

A very general discussion of the detection and extraction of signals in the presence of noise. Optimality is defined in terms of minimum risk or minimum loss of

information. Using the Bayes and minimax rules the authors show how the problem can be expressed in a natural way as a case in the theory of decision functions. In this way it is possible to discuss the optimality properties of receivers within the framework of this general theory. To exemplify this a number of special cases are considered. *U. Grenander* (Providence, R.I.).

Peterson, W. W.; Birdsall, T. G.; and Fox, W. C. The theory of signal dectectability. Trans. I.R.E. **PGIT-4** (1954), 171–212.

Considering the problem of detection of signals from the point of view of testing statistical hypotheses the authors discuss various ways of defining an optimal receiver. It is shown how one is led to receivers using a likelihood ratio as criterion for the presence of a signal both for finite sample size and in the sequential case. This is applied to the situation when noise is additive and consists of a bandlimited, Gaussian, stationary process. The paper contains a clear discussion in simple terms of various ensembles of signals. These include the following cases: 1) signal known except perhaps for the phase; 2) signal is itself a Gaussian process; 3) pulse train with incoherent phases; and 4) signal is one of an orthogonal set of functions with the phases known or random. The receiver operating characteristic is computed and represented graphically. *U. Grenander.*

★**Shannon, Claude E. The zero error capacity of a noisy channel.** Institute of Radio Engineers, Transactions on Information Theory, IT–2, September, 1956, pp. 8–19.

The author first considers finite, discrete, memoryless channels, i.e., those specified by an input-to-output transition matrix $\|p_i(j)\|$ of probabilities $p_i(j)$ with successive operations being independent. The zero error capacity, C_0, for a channel is defined as the upper bound of the rates of transmission using block codes subject to the restriction that the probability of error be zero. Methods are found for determining C_0 in special cases, and certain useful inequalities are proved. Results are then obtained for the values of C_0 and C (ordinary channel capacity) for channels with memory, i.e., channels with a feedback link. *R. A. Leibler.*

Bech, Niels Ivar. Deduction of a simple relationship between traffic offered and loss probabilities for the separate traffic sources. Teleteknik **1** (1957), 66–71.

The author of this paper has given a relation useful for control purposes when one uses electronic calculators in connection with a traffic generator as a traffic machine. With this relation one can control whether a traffic generator has produced Poisson distributed traffic or whether the lost call frequency measured in the artificial experiment is reliable. *A. Jensen* (Copenhagen).

Takács, L. On the generalization of Erlang's formula. Acta Math. Acad. Sci. Hungar. **7** (1956), 419–433. (Russian summary)

The author has shown 6 theorems, which give some very important generalizations of Erlang's classical formula on the probability for k wires out of m being occupied in a telephone exchange under certain conditions for the distributions of calls and exponentially distributed service time. It seems as if the author (p. 430.1.) has not been aware of the work by Conny Palm [Ericsson Technics no. **4** (1938), 39–58], who has worked with generalizations of Erlang's formula with Poisson in-put and generally distributed out-put. *A. Jensen.*

von Sydow, Lennart. Some aspects on the variations in traffic intensity. Teleteknik **1** (1957), 58–64.

In the paper the author has given a survey of the different characteristics of telephone traffic. In this connection he has mentioned Conny Palm's concepts "Träge Intensitätschwänkungen" and "Schnelle Intensitätschwänkungen" [Ericsson Technics no. **44** (1943)]. To these the author has added a discussion for the case where there is a trend in the intensity, and he has mentioned methods for its calculation. *A. Jensen.*

Harkevič, A. A. A problem. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 184. (Russian)

A problem, stated to be of practical interest in coding, is proposed in various formulations; e.g., given a fixed vertex of an n-dimensional cube and a positive number l, calculate the function $N(n, l)$ denoting the number of vertices whose distance from one another and from the given vertex is not less than l.

Webb, E. L. R. On the distribution of the product of diode detector waveforms. Canad. J. Phys. **34** (1956), 679–691.

The author finds the distributions of the products of certain random variables which are associated with the outputs of diode detectors when the inputs are signals and noise having certain specified forms.

R. A. Leibler (Silver Spring, Md.).

See also: Computing Machines: Tasman. **Probability:** Fortet and Canceill.

Control Systems

Oldenburger, Rufus. Optimum nonlinear control. Trans. A.S.M.E. **79** (1957), 527–542, discussion 542–546.

The author studies differential equations connected with the problems and variations of problems of the following type: if the speed setting of an engine governor is instantly switched to a new value find the optimal control, i.e. the return to equilibrium with minimal over and underswing, minimal duration etc. It is assumed that the rate of change of the controlling variable c is symmetrically bounded, $|c'| \leq k$, and that the max. and min. rate of change can be obtained arbitrarily. One of the main mathematical results is an interesting elementary proof which shows that under fairly general assumptions the optimum control is obtained if we let c' take only the values $\pm k$ and 0. The control functions so obtained are nonlinear. *J. Hartmanis* (Columbus, Ohio).

Brogle, A. P. On the frequency spectrum generated by a maneuvering aircraft. J. Aero. Sci. **24** (1957), 487–494.

With a view to applications to tracking systems, upper bounds for coefficients of Fourier series representations of movements of aircraft are obtained. *S. K. Zaremba.*

Poritsky, H. Hunting due to lost motion. Trans. A.S.M.E. **79** (1957), 577–584.

For a servomechanism designed to make the output θ follow the input I, with error $\delta = I - \theta$, the author studies the hunting motion due to lost motion, $\Delta x = 2h$, in one of the mechanical links. The system normally is assumed to satisfy equations of the form $p^2\theta = Ax$, $(p+B)x = (Cp+D)\delta$, where A, B, C, D are constants and $p = d/dt$; but with the lost motion it is assumed that $p^2\theta = A(x \pm h)$ according as x is decreasing or increasing. The case $I = 0$, in which there are no input signals, is studied relative to an initial displacement away from the quiescent position. This involves the analysis of a third-order differential system. The second-order system, in which $p^2\theta = Ax$ is replaced by $p\theta = Ax$, also is considered. It

is shown that, under certain circumstances, a periodic hunting motion always exists, and that the system, no matter how it started, will converge to this hunting motion. *E. F. Beckenbach* (Los Angeles, Calif.).

Markus, L.; and Wagner, N. Stability theorems for a class of nonlinear servomechanisms. J. Math. Mech. **6** (1957), 393–399.

The asymptotic behaviour of a class of non-linear servomechanisms which includes a spring whose restoring force is a cubic polynomial is investigated. A weak stability is also defined and studied for this class. *C. Saltzer* (Cleveland, Ohio).

Klotter, Karl. How to obtain describing functions for nonlinear feedback systems. Trans. A.S.M.E. **79** (1957), 509–512, discussion 512.

This is essentially an equivalent linearization technique. The systems in question are fed by harmonic forces and are described by nonlinear differential equations. Transfer functions are calculated as for linear systems except that the nonlinear terms in the differential equations are replaced by their averages over a quarter period of the driving frequency. *E. Pinney.*

See also: Algebra: Séminaire Dubreil et Pisot. **Mechanics of Particles and Systems:** Sergeev.

HISTORY, BIOGRAPHY

★ **Ал-Каши, Джемшид Гиясэддин. [Al-Kaši, Džemšid Giyasèddin.] Ключ арифметики: Трактат об окружности. [Key to arithmetic: Treatise on arcs of circles.] Translated from the Arabic by B. A. Rozenfel'd with commentaries by A. P. Yuškevič and B. A. Rozenfel'd. With reproductions of the Arabic manuscripts of both treatises.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 566 pp. 21.50 rubles.

Jamshid al-Kashi's floruit was the early part of the fifteenth century; he was an astronomer and mathematician connected with the court of Ulugh Beg at Samarkand and may have died in 1429. His book on the circle and his "Key to Arithmetic" have become better known in recent years through the work of the late P. Luckey, who gave a German translation with commentary of the book on the circle [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1950**, no. 6; MR **14**, 1051], and discussed the "Key" in "Die Rechenkunst bei Ġamšīd b. Mascūd al-Kāšī" [Deutsche Morgenländische Gesellschaft, Wiesbaden, 1951; MR **13**, 611] and in Math. Ann. **120** (1948), 217–274 [MR **9**, 484]. The present Russian edition of al-Kashi contains photo reprints of Arabic manuscripts of the two books, a Russian translation and an extensive commentary. Added is a fragment from Mariam Čelebi's "Rules for the construction and connection of tables": this author also worked at the Samarkand observatory and died in 1524/25. The Russian translation of this article is from the French version, published by L. Sédillot [Journal asiatique (5) **2** (1856), 333–356].

To those who have not read Luckey's writings, al-Kashi's works contain much which is surprising. In the treatise on the circle he computes, with essentially Archimedean methods, but using sexagesimal fractions, the value of 2π as 6, 16, 59, 28, 1, 34, 51, 46, 14, 50 for radius 60 (this means $6\times60+16+59\times60^{-1}+$etc.). He then converts it into a decimal fraction for radius 1 and obtains 2π in 16 decimals exact. Memory is aided by an Arabic distych for both values and an additional Persian one for the decimal expression.

In the "Key to Arithmetic" al-Kashi shows how to perform the elementary operations on integers and fractions (he writes $1\,\frac{7}{2}$ for $7\frac{1}{2}$), and has much to say on sexagesimal fractions. But he also uses decimal fractions with ease, illustrating how to multiply 25.07 into 14.3, obtaining 358.501 exactly in the same way as when multiplying integers. He indicates the integer part by black, the fractional part by red numerals (most of these arabic numerals are shaped somewhat differently from ours). He has tables to convert decimal ("Indian") notation into the sexagesimal one. Then he develops the binomial expression for general positive integral exponents with the general rule for the formation of the coefficients. This expression may already have been known to Omar Khayyam [see his "Algebra", p. 54 of the Kasir edition, Columbia Univ., New York, 1931], but Omar's text has not been found and al-Kashi seems the first text so far discovered which has the formula for $(a+b)^n$. This formula is used for the extraction of roots of higher order, and this leads to that numerical solution of higher degree equations now known as Horner's, or Horner-Ruffini's method [see Luckey's paper in Math. Ann. **120**]. The idea is not al-Kashi's, but appears already in the work of Chinese mathematicians of the Sung period [13th century, see L. Wang and J. Needham, T'oung Pao **43** (1955), 349–356, or Y. Mikami, Development of mathematics in China and Japan, Teubner, 1913]. There is more of interest in the "Key", especially for the historian of trigonometry and for the historian of architecture. Al-Kashi's method of solving a cubic equation by iteration (and in sexagesimal fractions) is commented upon in the commentary to the Čelebi article, which deals with methods of constructing sine tables; al-Kashi's method leads to the value of sin 1° accurate up to sexagesimal quarts. *D. J. Struik.*

Il'in, A. S. The philosophical significance of the geometry of N. I. Lobačevskiĭ. Mat. v. Škole 1957, no. 3, 1–4. (Russian)

A discussion of the role played by Lobačevskiĭ's geometry in investigations of the connection between space and matter.

Anonymous. On the 250th anniversary of the birth of Leonard Euler. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 3–9. (Russian)

Mihailov, G. K. On the arrival of Leonard Euler in Petersburg (From an early correspondence of L. Euler with D. Bernoulli and from other sources). Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 10–37. (Russian)

Reichardt, Hans. Gauss' Arbeiten über Algebra und Zahlentheorie. S.-B. Berlin. Math. Ges. **1954/55–1955/56**, 11–23.

Levi, F. W. Gauss und das Raumproblem. S.-B. Berlin. Math. Ges. **1954/55–1955/56**, 24–29.

Aiton, E. J. The contributions of Newton, Bernoulli and Euler to the theory of the tides. Ann. of Sci. **11** (1956), 206–223.

Aiton, E. J. Descartes's theory of the tides. Ann. of Sci. **11** (1955), 337–348 (1957).

See also: Differential Geometry: Alexits.

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